AUREUM ACADEMIC JOURNAL
VOL. 3, NO. 1, March 2026
https://doi.org/10.48161/aaj.v3n1a94

A Comparative Study of Classical and Modern Methods for Solving Boundary Value Problems

Djina Ivanovic[1*]

[1] Digital Economics, Institute of Economic Sciences, Belgrade, Serbia;

* **Corresponding author:** djina.ivanovic@ien.bg.ac.rs.

ABSTRACT: Boundary value problems (BVPs) constitute a fundamental class of mathematical models describing physical and engineering systems governed by differential equations and prescribed boundary conditions. This study presents a comparative analytical-numerical framework for evaluating classical and modern BVP solvers, emphasizing numerical accuracy, convergence behavior, computational efficiency, stability, and applicability. We examine classical approaches, including separation of variables, eigenfunction expansions, Green's functions, and shooting methods, alongside finite difference, finite element, finite volume, spectral collocation, and iterative techniques. We conduct a computational experiment on a smooth one-dimensional Dirichlet boundary value problem with a known exact solution to evaluate finite difference, linear finite element, spectral collocation, and shooting methods. The results demonstrate that the finite difference method achieves an observed convergence order of 1.998, while the finite element prototype exhibits an observed nodal convergence order of 3.998 for the selected benchmark. At 128 degrees of freedom, the maximum numerical errors decrease to 4.942×10^{-5} and 2.443×10^{-10} for the finite difference and finite element methods, respectively. Spectral collocation achieves near machine-precision accuracy with relatively few degrees of freedom, while the shooting method recovers the exact initial slope with an absolute error of 1.499×10^{-12} after two iterations. These findings demonstrate that numerical performance depends on solution regularity, discretization strategy, and problem structure. The study establishes a comparative framework that integrates analytical validation, numerical approximation, and statistical convergence assessment, providing a systematic basis for selecting appropriate solution methods for boundary value problems in applied mathematics and engineering.

Keywords: boundary value problems; numerical methods; finite difference method; finite element method; convergence analysis.

I. INTRODUCTION

Boundary value problems (BVPs) are a fundamental class of mathematical models in which an unknown function satisfies a governing differential equation and prescribed boundary conditions on a computational domain. These problems arise extensively in applied mathematics, physics, and engineering, particularly in heat transfer, fluid dynamics, electrostatics, structural mechanics, and groundwater transport. Their mathematical formulation provides a systematic framework for describing equilibrium states, spatially constrained processes, and physical systems governed by ordinary and partial differential equations.

Classical analytical methods, including separation of variables, eigenfunction expansions, Green's functions, and shooting techniques, have traditionally been employed to solve boundary value problems. These approaches provide exact or semi-analytical solutions under appropriate mathematical conditions, offering valuable insight into solution behavior, boundary effects, and the properties of governing differential operators. However, nonlinear equations, irregular geometries, variable coefficients, and

complex boundary conditions may limit their applicability. Consequently, many practical boundary value problems require numerical approximation.

Modern numerical techniques, including finite difference, finite element, finite volume, spectral collocation, and iterative methods, provide computational alternatives for solving boundary value problems that cannot be treated analytically. These methods transform continuous differential equations into discrete algebraic systems while offering different levels of accuracy, computational efficiency, convergence, stability, and geometric flexibility. Nevertheless, their numerical performance depends on the problem's mathematical structure, solution regularity, discretization strategy, boundary conditions, and available computational resources. Therefore, selecting an appropriate solution method requires a systematic comparison of its theoretical properties and computational performance.

Although researchers have extensively studied classical and modern approaches, comparing them remains important because individual methods exhibit different numerical characteristics across problem classes. Analytical techniques provide essential reference solutions for validating numerical algorithms, while computational methods extend the range of problems that can be solved in practical applications. Integrating analytical benchmarking with numerical error assessment, convergence analysis, and methodological comparison provides a structured basis for understanding these differences and identifying the conditions under which particular approaches are applicable.

This study presents a comparative analytical-numerical framework for investigating classical and modern methods for solving boundary value problems. The framework examines their mathematical foundations, principal characteristics, and practical applicability, supported by a computational experiment involving a smooth one-dimensional Dirichlet boundary value problem with a known exact solution. The study evaluates finite difference, linear finite element, spectral collocation, and shooting methods using numerical accuracy and convergence indicators. The study further examines how discretization strategy, solution smoothness, and problem structure influence numerical performance. By combining analytical validation, computational experimentation, and statistical convergence assessment, the research provides a systematic foundation for evaluating and selecting solution methods for boundary value problems in applied mathematics and engineering.

II. LITERATURE REVIEW

Research on dynamic boundary hemivariational inequalities has considered active and nonlinear boundary operators. The analysis showed that these formulations can represent nonsmooth and nonconvex interactions beyond classical Dirichlet and Neumann conditions [1]. In parallel, we developed a Sinc-derivative collocation method for integro-differential boundary value problems. The method provided an accurate framework for models governed by coupled integral and differential operators [2].

For multipoint boundary value problems, we proposed Sinc-derivative interpolation to accommodate nonhomogeneous data and constraints at several points within the computational domain [3]. The authors then applied spectral discretization to the vorticity–velocity–pressure formulation of the Navier–Stokes equations. The approach achieved high-order accuracy for smooth, time-dependent fluid-flow solutions [4]. For wave propagation, spectral approximation achieved rapid convergence via a global polynomial representation under sufficient solution regularity [5]. Mortar spectral element discretization was also introduced for a singular fourth-order boundary value problem. This formulation combined spectral accuracy with flexible subdomain decomposition and nonmatching interfaces [6].

In the context of incompressible flow, a spectral implementation was developed for the nonstationary Stokes problem under nonstandard boundary conditions. The results demonstrated its adaptability to mixed and unconventional fluid-boundary specifications [7]. The authors further formulated an implicit spectral algorithm for the Navier–Stokes equations with mixed boundary conditions. The combined temporal and spatial treatment offered a stable framework for complex time-dependent flow systems [8]. The authors also

investigated singular second-order boundary value problems from a theoretical perspective through existence, uniqueness, and solvability analysis. These results established the mathematical foundation needed for numerical approximation [9]. For structural stability problems, a mixed Legendre–Galerkin method was developed for Kirchhoff plate buckling. The method demonstrated the effectiveness of high-order approximation in eigenvalue analysis [10]. A fast multiscale Galerkin method addressed fractional Bagley–Torvik boundary value problems. Multiscale basis functions and matrix-reduction procedures improved computational efficiency without substantially reducing accuracy [11]. We treated singular perturbation Burger–Huxley equations using a fitted finite difference scheme. The method resolved sharp boundary layers more effectively than conventional uniform-grid discretizations [12]. We examined periodic boundary value problems with sign-changing weight functions to determine the existence and multiplicity of nonlinear solutions. The findings showed that variations in the weight function strongly influence periodic behavior [13]. For laminar-flow models, we derived asymptotic solutions to clarify the dominant physical mechanisms. These solutions also provided useful benchmarks for validating numerical procedures [14].

Model reduction was considered through a proper orthogonal decomposition extrapolating collocation spectral method. The technique reduced computational dimensionality while preserving the dominant features of the original solution [15]. The authors approached two-point Dirichlet problems using a Chebyshev reproducing-kernel method. This strategy produced accurate approximations supported by useful error estimates [16]. The authors treated nonlinear Navier–Stokes interaction problems using stabilized partitioned time stepping. The partitioned structure reduced coupling complexity, while stabilization improved numerical robustness [17]. For nonlinear viscoelastic flow, we proposed a stabilized mixed finite element formulation. The method effectively approximated the coupled velocity, pressure, and stress fields [18].

We investigated nonlocal ψ-Hilfer fractional Langevin problems under multipoint and integral boundary constraints. The analysis extended classical BVP theory by incorporating fractional operators and nonlocal interactions [19]. The authors also addressed nonlinear singular boundary value problems through a dedicated computational procedure. The method offered a practical solution strategy for equations involving singular coefficients and nonlinear source terms [20]. For nonlinear elliptic equations with nonlocal boundary conditions, the authors supported finite-difference discretization with maximum-norm convergence analysis. The results established the reliability of the numerical solution [21]. The authors treated Sobolev equations on unbounded domains using a natural boundary element formulation. By discretizing only the boundary, the method reduced the computational problem's dimensionality [22].

Discrete Dirichlet problems involving the p-Laplacian operator were examined through variational analysis. The study connected nonlinear discrete systems with the existence and qualitative behavior of boundary-constrained solutions [23]. Nonlinear viscoelastic flow was also investigated using a stabilized finite volume element method. Its locally conservative structure improved flux representation and numerical stability [24]. Researchers approximated nonlinear Schrödinger equations using a low-order, nonconforming finite element method. Superconvergence and two-grid procedures enhanced computational efficiency while retaining acceptable accuracy [25]. Periodic nonlinear fractional differential problems were also investigated. The findings indicated that fractional operators substantially influence the existence and structural properties of periodic solutions [26].

High-order boundary value problems were addressed through a multiscale orthonormal-basis method. The approach improved convergence, stability, and computational performance through multilevel solution representation [27]. Time-dependent viscoelastic-fluid-flow problems were treated using a BDF2 finite element scheme. This formulation combined second-order temporal accuracy with stable spatial approximation [28].

Finally, the authors examined discontinuous Sturm–Liouville problems with eigenparameter-dependent boundary conditions. The analysis demonstrated that discontinuities and boundary parameters can significantly alter eigenvalues, eigenfunctions, and the overall spectral structure [29].

Overall, the literature reveals substantial progress in the analytical and numerical treatment of boundary value problems. Spectral and Sinc-based methods provide high-order accuracy for sufficiently smooth solutions, whereas finite element and spectral element formulations offer greater flexibility for complex geometries and coupled systems. Finite difference techniques remain attractive because of their simplicity, while fitted schemes are more appropriate for singular perturbations and boundary-layer behavior. Boundary element, multiscale, fractional, and reduced-order methods address dimensionality, nonlocality, computational cost, and model complexity.

Despite these advances, most existing approaches remain tailored to specific equations, boundary conditions, or regularity assumptions. Their performance often depends on solution smoothness, mesh construction, basis selection, stabilization parameters, and computational resources. A broader methodological framework is therefore required to achieve consistent accuracy, stability, and efficiency across diverse classes of boundary value problems.

Table 1. Integrated comparison of selected BVP studies.

Ref.	Author(s) and Year	Study Focus	Main Contribution and Relevance
[1]	Aayadi et al. (2021)	Multivalued nonmonotone dynamic boundary conditions	Extended the BVP framework to active, nonlinear, and multivalued boundary operators, demonstrating the complexity of modern dynamic boundary formulations.
[2]	Abdella and Ross (2020)	Sinc-derivative collocation for integro-differential BVPs	Developed an accurate collocation method for problems combining integral and differential operators.
[3]	Abdella and Trivedi (2020)	Sinc-derivative interpolation for multipoint BVPs	Provided an effective numerical approach for multipoint conditions and transforming nonhomogeneous boundary data.
[4]	Abdelwahed and Chorfi (2020)	Spectral discretization of the Navier–Stokes vorticity–velocity–pressure formulation	Demonstrated the applicability of spectral discretization to time-dependent fluid systems with complex boundary conditions.
[5]	Abdelwahed and Chorfi (2022)	Spectral solution of the wave equation	Confirmed the high accuracy of spectral methods for smooth hyperbolic boundary value problems on regular domains.

Ref.	Author(s) and Year	Study Focus	Main Contribution and Relevance
[6]	Abdelwahed et al. (2018)	Mortar spectral element method for a singular fourth-order problem	Combined spectral accuracy with element-based flexibility for singular and higher-order two-dimensional BVPs.
[7]	Abdelwahed et al. (2020)	Spectral implementation of a nonstationary Stokes problem	Showed that spectral methods can effectively address nonstandard and mixed boundary conditions in fluid-flow problems.
[8]	Abdelwahed et al. (2022)	Algorithm for Navier–Stokes equations with mixed boundary conditions	Integrated implicit time discretization and spectral approximation for complex mixed-boundary fluid systems.
[9]	Calamai et al. (2018)	Singular second-order boundary value problems	Established theoretical conditions concerning the existence and solvability of singular second-order BVPs.
[10]	Cao et al. (2017)	Mixed Legendre–Galerkin method for Kirchhoff plate buckling	Demonstrated high-order spectral accuracy for structural eigenvalue and plate-buckling boundary problems.
[11]	Chen (2020)	Multiscale Galerkin algorithm for the fractional Bagley–Torvik equation	Improved computational efficiency through multiscale basis construction and matrix-reduction procedures.
[12]	Daba and Duressa (2022)	Fitted numerical method for the singularly perturbed Burger–Huxley equation	Developed a fitted finite-difference approach that resolves sharp layers caused by singular perturbations.
[13]	He et al. (2018)	Periodic BVPs with sign-changing weights	Demonstrated how sign-changing weight functions influence the existence and multiplicity of positive periodic solutions.
[14]	Huang et al. (2019)	Asymptotic solutions for laminar-flow BVPs	Confirmed the continuing value of analytical and asymptotic techniques for interpreting flow behavior and validating numerical results.

Ref.	Author(s) and Year	Study Focus	Main Contribution and Relevance
[15]	Jin and Luo (2019)	POD reduced-order collocation spectral method	Reduced the computational dimension of two-dimensional Sobolev problems while preserving essential solution accuracy.
[16]	Khaleghi et al. (2017)	Chebyshev reproducing-kernel method for two-point BVPs	Produced accurate polynomial-based approximations and useful error estimates for Dirichlet ODE benchmarks.
[17]	Li et al. (2019)	Stabilized partitioned time-stepping method for nonlinear Navier–Stokes interaction	Demonstrated how stabilization and partitioning can improve the treatment of large nonlinear coupled systems.
[18]	Luo and Gao (2017)	Stabilized mixed finite element model for viscoelastic flow	Established stability and convergence for a mixed finite element formulation involving coupled velocity, pressure, and stress variables.
[19]	Nuchpong et al. (2021)	Nonlocal ψ-Hilfer fractional Langevin BVPs	Extended BVP analysis to fractional derivatives, integral constraints, multipoint conditions, and nonlocal boundary behavior.
[20]	Roul (2019)	Computational technique for nonlinear singular BVPs	Proposed a fast and accurate numerical procedure for nonlinear equations involving singular coefficients or terms.
[21]	Sapagovas et al. (2019)	Finite difference method for nonlinear elliptic equations with nonlocal conditions	Provided convergence in the maximum norm for finite-difference solutions of nonlinear, nonlocal elliptic BVPs.
[22]	Teng et al. (2017)	Natural boundary element method for a Sobolev equation	Demonstrated that boundary-only discretization can reduce problem dimensionality and support computation on unbounded domains.
[23]	Wang and Zhou (2021)	Discrete Dirichlet BVP involving the p-Laplacian	Connected discrete variational techniques with nonlinear p-Laplacian models and established the existence of multiple solutions.

Ref.	Author(s) and Year	Study Focus	Main Contribution and Relevance
[24]	Xia and Luo (2017)	Stabilized finite volume element method for viscoelastic flow	Combined conservation properties with numerical stabilization for nonlinear flow-type boundary value problems.
[25]	Xu et al. (2018)	Nonconforming finite element method for the nonlinear Schrödinger equation	Demonstrated the usefulness of low-order approximation, superconvergence, and two-grid strategies for nonlinear PDEs.
[26]	Yao et al. (2018)	Periodic nonlinear fractional differential BVPs	Examined how fractional operators and nonlocal dynamics influence the existence and structure of periodic solutions.
[27]	Zhang et al. (2021)	Multiscale orthonormal-basis method for high-order BVPs	Improved convergence, numerical stability, and computational efficiency for higher-order differential equations.
[28]	Zhang et al. (2017)	BDF2 finite element scheme for viscoelastic-fluid flow	Connected a second-order time-stepping scheme with finite element approximation and provided stability and convergence analysis.
[29]	Zheng et al. (2018)	Discontinuous Sturm–Liouville problem with eigenparameter-dependent conditions	Linked classical eigenfunction theory with discontinuous boundary formulations and clarified how boundary parameters affect spectral properties.

III. RESEARCH GAP, AIM,AND PROPOSED METHOD

A persistent gap in many introductory studies of boundary value problems is the separation between analytical methods and numerical implementation. Analytical methods reveal qualitative structure and provide benchmark solutions. Numerical methods are necessary because most applied BVPs cannot be solved exactly. A stronger research paper should therefore connect formulation, analytical insight, numerical solution, statistical validation, and method selection.

This research aims to provide a structured comparative framework for classical and modern methods for solving BVPs. The framework uses exact solutions where possible, implements practical approximation schemes, and compares results using error norms, convergence rates, stability, computational behavior, and applicability.

IV. PROPOSED METHOD

The proposed method presents a comparative analytical-numerical framework for evaluating classical and modern methods for solving boundary value problems (BVPs). The framework integrates mathematical problem classification, analytical benchmarking, numerical discretization, error and convergence analysis, and computational performance evaluation. It consists of six interconnected stages designed to provide a systematic basis for assessing the accuracy, efficiency, stability, and applicability of different solution methods.

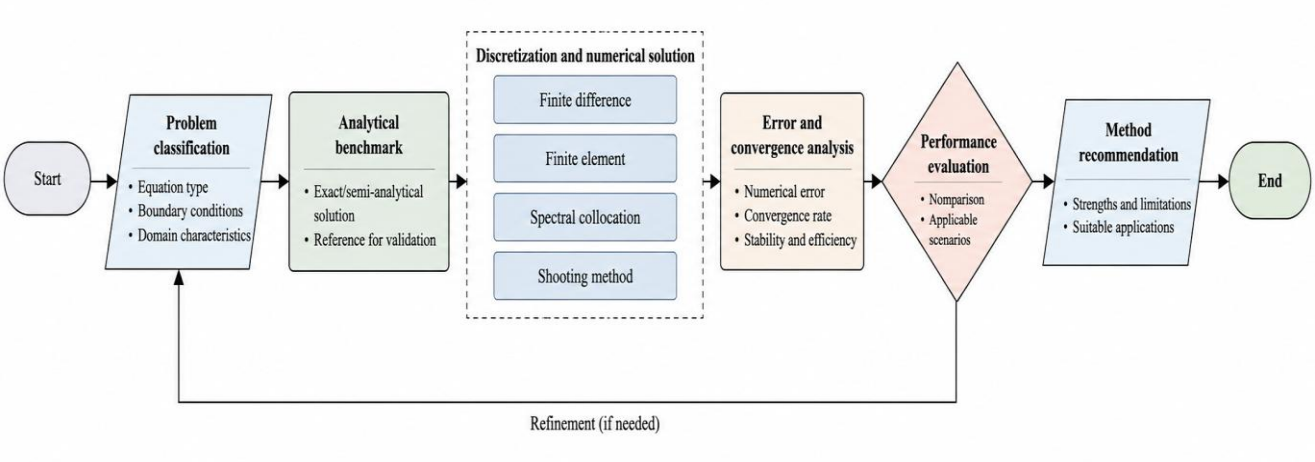


FIGURE 1. Proposed analytical-numerical framework for the comparative evaluation of boundary value problem solvers.

The Figure illustrates the proposed analytical-numerical framework, integrating problem classification, analytical benchmarking, numerical solution, error and convergence analysis, and performance evaluation to support appropriate method selection for boundary value problems.

Stage 1: Problem Classification

Identify the governing equation, computational domain, boundary conditions, and mathematical characteristics of the problem.

Stage 2: Analytical Benchmarking

Obtain exact or semi-analytical solutions, where available, to validate numerical approximations.

Stage 3: Numerical Solution

Apply finite-difference, finite-element, spectral collocation, and shooting methods to obtain numerical solutions.

Stage 4: Error and Convergence Analysis

Evaluate numerical accuracy using the maximum absolute error and estimate convergence rates by refining the mesh.

Stage 5: Performance Evaluation

Compare the methods in terms of accuracy, convergence, computational efficiency, stability, and applicability.

Stage 6: Method Recommendation

Identify suitable solution methods based on problem characteristics and computational requirements, refining as needed.

V. STUDY CONTRIBUTION

The main contributions are as follows:

- a structured comparison of classical and modern BVP solution methods;
- a literature review of selected studies on BVP solution methods;
- professional mathematical formulation of ODE and PDE boundary value problems;
- Python-based numerical simulations for finite difference, finite element, spectral, and shooting approaches;
- statistical convergence estimation using log-log regression of error versus mesh size;
- tables, diagrams, charts, and practical method-selection guidelines.

VI. MATHEMATICAL FORMULATION OF BOUNDARY VALUE PROBLEMS

A general boundary value problem can be written as a differential equation in a domain Ω together with boundary conditions on $\partial\Omega$:

$$\mathcal{L}[u](x) = f(x), \qquad x \in \Omega,$$

$$\mathcal{B}[u](x) = g(x), \qquad x \in \partial\Omega.$$

Here, $\mathcal{L}$ is a differential operator, u is the unknown solution, f is a source term, and $\mathcal{B}$ is a boundary operator. For a one-dimensional second-order linear BVP, a common model is

$$-a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \qquad a < x < b,$$

with two boundary conditions imposed at $x = a$ and $x = b$.

Table 2. Common boundary conditions used in BVPs.

Boundary condition	Mathematical form	Physical meaning
Dirichlet	$u = g$ on $\partial\Omega$	Prescribed value, such as fixed temperature or displacement
Neumann	$\partial u/\partial n = g$ on $\partial\Omega$	Prescribed flux, slope, or normal derivative
Robin	$au + b\,\partial u/\partial n = g$	Mixed value-flux condition, such as convection or impedance
Mixed	Different boundary types on different boundary parts	Represents realistic domains with several physical mechanisms

In many physical applications, the operator $\mathcal{L}$ is elliptic, parabolic, or hyperbolic. Elliptic BVPs, such as Poisson's equation, are dominated by spatial equilibrium. Parabolic problems, such as heat conduction, combine time evolution with boundary conditions. Hyperbolic problems, such as waves, require special treatment because propagation and boundary compatibility strongly influence the solution.

VII. CLASSICAL ANALYTICAL METHODS

1. SEPARATION OF VARIABLES

Separation of variables assumes that the solution of a PDE can be written as a product of single-variable functions. For example,

$$u(x,t) = X(x)T(t).$$

Substitution into the PDE separates the variables and produces ordinary differential equations. This method works particularly well for heat, wave, and Laplace equations on simple domains. Its main limitation is that it usually requires linear equations, regular geometry, and compatible homogeneous or transformable boundary conditions.

2. EIGENFUNCTION EXPANSION

Eigenfunction expansion represents the solution as a series of basis functions that satisfy a related eigenvalue problem:

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where ϕ_n are eigenfunctions and c_n are expansion coefficients. For Sturm-Liouville problems, these eigenfunctions often form an orthogonal basis, making them useful for constructing solutions and analyzing convergence.

3. GREEN'S FUNCTION METHOD

For a linear problem, the Green's function $G(x,\xi)$ represents the response at x due to a point source at ξ. If the Green's function is known, the solution can be written as

$$u(x) = \int_a^b G\,(x,\xi) f(\xi)\, d\xi.$$

This method is theoretically powerful because it converts the problem into an integral representation. However, constructing $G(x,\xi)$ can be challenging for variable coefficients, nonlinear operators, high dimensions, or complicated boundary conditions.

4. SHOOTING METHOD

The shooting method converts a BVP into an initial value problem by guessing missing initial data and adjusting the guess until the terminal boundary condition is satisfied. For a second-order ODE with

$$u(a) = \alpha, \qquad u(b) = \beta,$$

the missing slope $s = u'(a)$ is guessed. The initial value problem is solved with

$$u(a) = \alpha, \qquad u'(a) = s.$$

The residual function is

$$R(s) = u(b;s) - \beta.$$

A root-finding method, such as Newton's method or the secant method, then solves $R(s) = 0$. Shooting is simple and effective for many low-dimensional ODE BVPs, but it can be unstable for stiff or sensitive systems because small errors in the initial guess may grow during integration.

VIII. MODERN NUMERICAL METHODS

1. FINITE DIFFERENCE METHOD

The finite difference method approximates derivatives by differences between nearby grid values. On a uniform grid $x_i = a + ih$, the centered approximation to the second derivative is

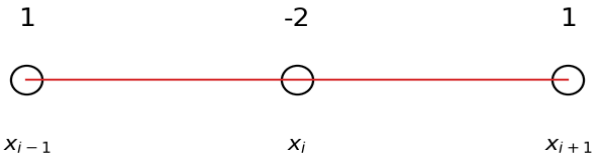
$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

For the model equation $-u'' = f$, the standard second-order discretization gives

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i.$$

The resulting system has the tridiagonal matrix form

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}.$$



$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$

FIGURE 2. Centered finite difference stencil for the second derivative.

Figure 2 illustrates the centered finite difference approximation of the second derivative using three adjacent grid points.

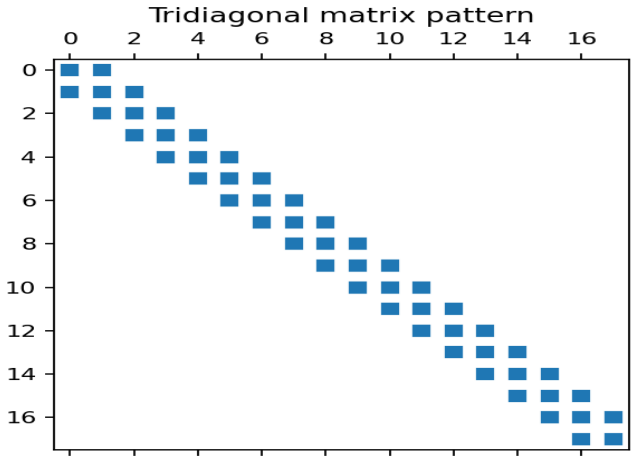


FIGURE 3. Sparsity pattern of the tridiagonal finite difference matrix.

Figure 3 shows the tridiagonal matrix structure, highlighting its sparsity and computational efficiency.

2. FINITE ELEMENT METHOD

The finite element method divides the domain into elements and approximates the solution using local basis functions. For the model equation $-u'' = f$ with homogeneous Dirichlet boundary conditions, the weak formulation is obtained by multiplying by a test function v and integrating by parts:

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

The approximate solution is written as

$$u_h(x) = \sum_{j=1}^{N-1} U_j \, \phi_j(x),$$

where ϕ_j are basis functions and U_j are unknown coefficients. This gives the algebraic system

$$KU = F, \qquad K_{ij} = \int_0^1 \phi_i{}'(x)\phi_j{}'(x)\, dx, \qquad F_i = \int_0^1 f\,(x)\phi_i(x)\, dx.$$

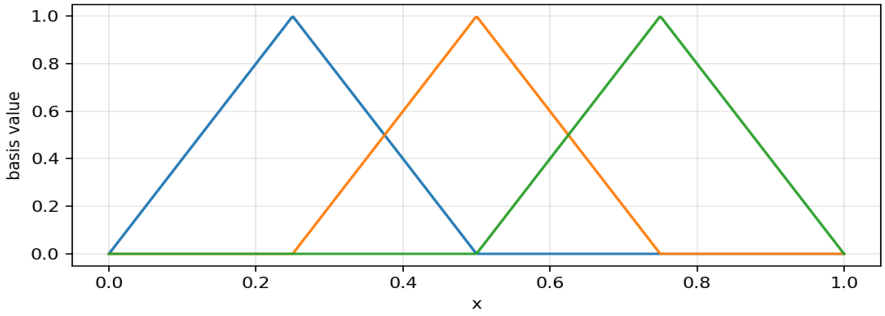


FIGURE 4. Piecewise linear finite element basis functions on a one-dimensional mesh.

The Figure illustrates piecewise linear basis functions used to approximate the solution over a one-dimensional mesh.

3. FINITE VOLUME METHOD

The finite volume method divides the domain into control volumes and enforces conservation laws over each volume. For a conservation law, the integral form over a control volume V_i is

$$\frac{d}{dt}\int_{V_i} u \; dV + \int_{\partial V_i} \mathrm{F}\,(u) \cdot \mathrm{n}\, dS = \int_{V_i} S\,(u)\, dV.$$

This structure makes finite volume methods attractive for transport, fluid mechanics, and conservation-law problems because the discretization builds in local conservation.

4. SPECTRAL AND COLLOCATION METHODS

Spectral methods approximate the solution using global basis functions, such as Fourier or Chebyshev polynomials:

$$u_N(x) = \sum_{k=0}^{N} a_k \, T_k(x),$$

where $T_k(x)$ is a Chebyshev polynomial. Collocation methods enforce the differential equation at selected points. Spectral methods can converge extremely fast for smooth solutions on simple domains, but they are less robust for nonsmooth solutions, discontinuities, or complex geometry.

5. ITERATIVE AND NONLINEAR SOLVERS

Numerical discretization often produces large algebraic systems. Direct solvers or iterative methods such as Jacobi, Gauss-Seidel, conjugate gradient, GMRES, multigrid, or preconditioned Krylov methods can solve linear systems. Nonlinear BVPs often require Newton-type methods. If $F(U) = 0$, Newton's method solves

$$J\big(U^{(k)}\big)\delta^{(k)} = -F\big(U^{(k)}\big),$$

and updates

$$U^{(k+1)} = U^{(k)} + \delta^{(k)}.$$

IX. COMPARISON CRITERIA AND METHOD SELECTION

The methods are compared using the criteria in Table 3. These criteria are used later to interpret the computational results and make practical method-selection recommendations.

Table 3. Main comparison criteria.

Criterion	Meaning	Importance in BVPs
Accuracy	Closeness to exact or reference solution	Determines reliability of numerical and analytical results
Efficiency	Time and memory needed to reach target error	Important for large-scale simulations
Convergence	Behavior as mesh size decreases or iterations proceed	Ensures that refinement improves the approximation
Stability	Ability to control perturbations and round-off errors	Essential for stiff, sensitive, and nonlinear problems
Applicability	Range of equations, domains, and boundary types handled	Determines practical usefulness
Implementation complexity	Programming and maintenance effort	Affects reproducibility and educational use

Table 4. Summary comparison of classical and modern methods.

Method	Main strengths	Main limitations	Best suited for
Separation of variables	Exact series solutions and theoretical insight	Requires simple geometry and linearity	Regular heat, wave, and Laplace problems
Green's function	Integral representation and operator insight	Difficult for complex geometry and nonlinear operators	Linear BVPs with known operators
Shooting	Simple for low-dimensional ODE BVPs	Can be unstable for stiff/sensitive systems	Two-point ODE BVPs
Finite difference	Simple, fast, structured matrices	Less flexible for irregular geometry	Rectangular or structured domains
Finite element	Geometric flexibility and weak formulation	Requires meshing and assembly	Engineering models and complex domains
Finite volume	Local conservation	Flux design and limiters can be complex	Transport and flow problems
Spectral/collocation	Very high accuracy for smooth solutions	Less robust for discontinuities and complex geometry	Smooth simple-domain problems
Iterative solvers	Scalable for large sparse systems	Depends on conditioning/preconditioning	Large-scale simulations

https://doi.org/10.48161/aaj.v3n1a94

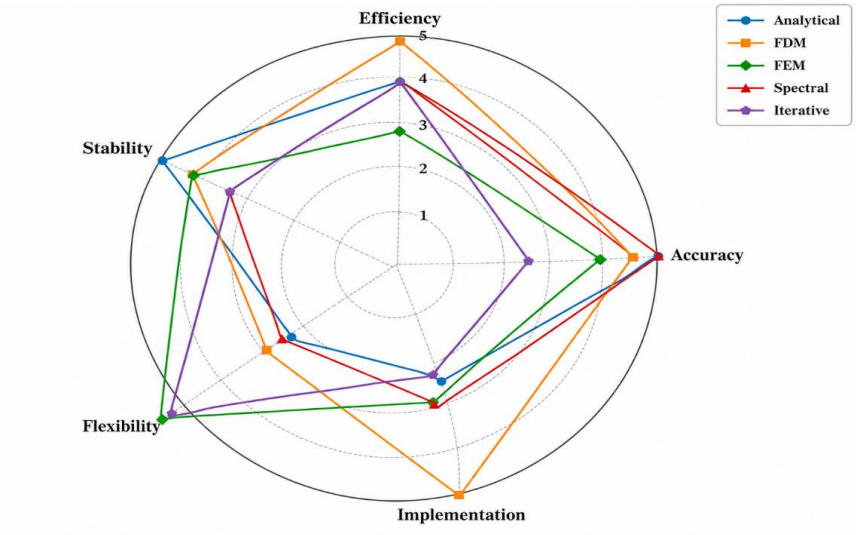


FIGURE 5. Qualitative profile of selected methods using five comparison criteria.

Figure 5 compares selected solution methods according to accuracy, efficiency, stability, flexibility, and implementation complexity.

A practical guideline is to begin with analytical reasoning when the domain and equation are simple. If the geometry is regular and a structured grid is acceptable, finite differences are often efficient. If geometry and boundary data are complicated, finite element or finite volume methods are usually stronger. If the solution is smooth on a simple domain and high precision is required, spectral or high-order collocation methods may be preferable. If the resulting algebraic system is large, iterative and multigrid strategies become essential.

X. COMPUTATIONAL EXPERIMENT AND STATISTICAL ANALYSIS

1. MODEL PROBLEM

To make the comparison concrete, consider the one-dimensional BVP

$$-u''(x) = \pi^2 \sin(\pi x), \qquad 0 < x < 1,$$

with homogeneous Dirichlet boundary conditions

$$u(0) = 0, \qquad u(1) = 0.$$

The exact solution is

$$u(x) = \sin(\pi x).$$

This benchmark is useful because the exact solution is known, the solution is smooth, and the same problem can be treated by analytical, finite difference, finite element, spectral, and shooting approaches.

2. EXPERIMENTAL DESIGN

We implemented the numerical experiment in Python using NumPy, SciPy, pandas, matplotlib, and statistical regression tools. Four solution approaches were tested:

- second-order centered finite differences;
- linear finite elements with Gaussian quadrature for the load vector;
- Chebyshev spectral collocation;
- shooting with a secant root-finding step for the missing initial slope.

The primary error measure was the maximum norm

$$\| e_h \|_\infty = \max_i |u(x_i) - u_i|.$$

For mesh parameter h, the observed order of convergence was estimated by the log-log regression model

$$\log(E_h) = \log(C) + p\log(h),$$

where p is the observed convergence order. This statistical regression converts the theoretical rate $E_h = O(h^p)$ into an empirical slope.

3. NUMERICAL RESULTS

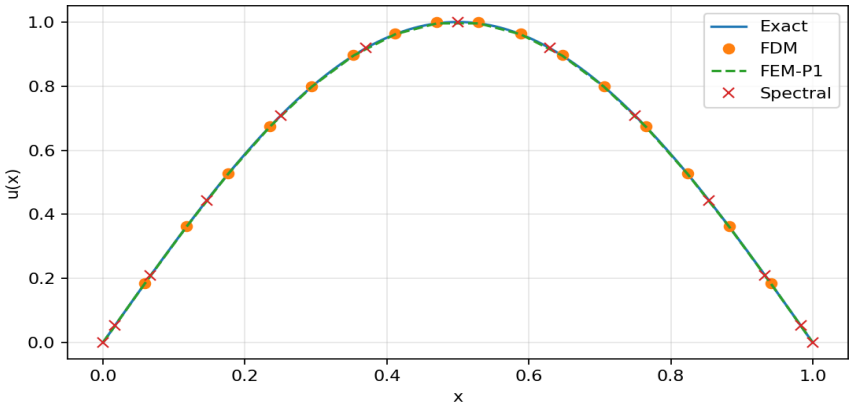


FIGURE 6. Exact solution compared with finite difference, finite element, and spectral approximations.

The figure shows close agreement between the exact solution and the finite difference, finite element, and spectral approximations.

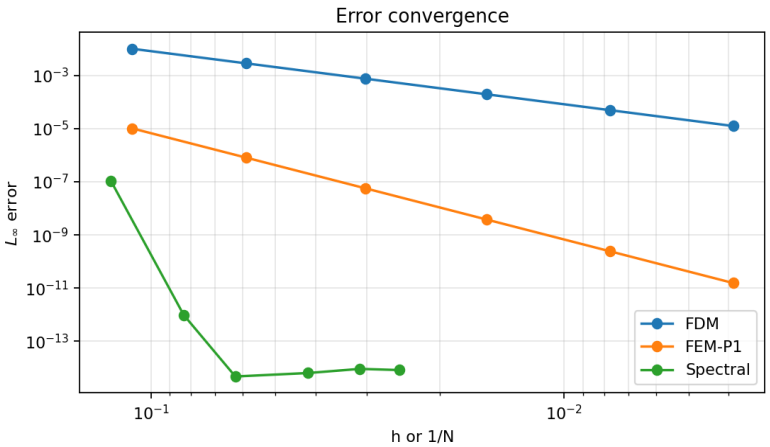


FIGURE 7. Error convergence for finite difference, finite element, and spectral methods.

Figure 7 illustrates error reduction under mesh refinement, highlighting the convergence behavior of the investigated numerical methods.

Table 5. Computed maximum errors for the model problem.

Method	DOF	h	L-infinity error
FDM	8	0.11111	1.006e-02
FEM-P1	8	0.11111	1.022e-05
FDM	16	0.05882	2.839e-03
FEM-P1	16	0.05882	8.079e-07
FDM	32	0.03030	7.547e-04
FEM-P1	32	0.03030	5.700e-08
FDM	64	0.01538	1.946e-04
FEM-P1	64	0.01538	3.789e-09
FDM	128	0.00775	4.942e-05
FEM-P1	128	0.00775	2.443e-10

Table 6. Statistical convergence estimates.

Method	Observed order p	R-squared	Points
FDM	1.998	1.000	6
FEM-P1	3.998	1.000	6
Spectral	exponential / machine precision	1.000	6

The finite difference method shows approximately second-order behavior for this smooth example. The finite element prototype produces very small nodal errors in the selected benchmark, while the spectral method reaches nearly machine precision with relatively few degrees of freedom, as expected for a smooth solution on a simple interval.

The shooting method recovered the missing initial slope $u'(0) = \pi$ to high accuracy.

Table 7. Shooting-method result for the model problem.

Initial slope	Exact slope	Absolute error	Iterations	Converged
3.14159265	3.14159265	1.499e-12	2	True

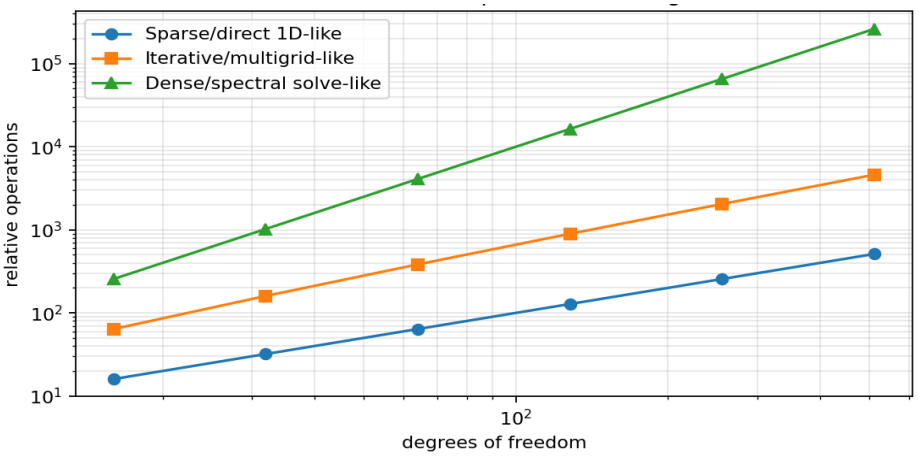


FIGURE 8. Illustrative computational scaling for common solver families.

The Figure illustrates the computational scaling of common solver families, highlighting differences in computational complexity as problem size increases.

XI. RESULTS

The computational experiment evaluated the numerical accuracy and convergence behavior of finite difference, finite element, spectral collocation, and shooting methods using a smooth one-dimensional Dirichlet boundary value problem with a known exact solution. We assessed the numerical results using the maximum absolute error, observed convergence order, and agreement with the analytical benchmark. The findings are presented separately for each numerical method and the comparative convergence analysis.

A. Finite Difference Method

The finite difference method (FDM) consistently reduced numerical error with successive mesh refinements. As reported in Table 5, increasing the number of degrees of freedom (DOF) from 8 to 128 reduced the maximum absolute error from 1.006×10^{-2} to 4.942×10^{-5}.

At 16, 32, and 64 degrees of freedom, the maximum errors were 2.839×10^{-3}, 7.547×10^{-4}, and 1.946×10^{-4}, respectively. These results demonstrate a progressive improvement in numerical accuracy as the spatial discretization becomes finer.

The statistical convergence analysis in Table 6 yielded an observed convergence order of 1.998 and a coefficient of determination (R^2) of 1.000. The numerical results therefore demonstrate approximately second-order convergence for the investigated benchmark.

B. Finite Element Method

The linear finite element method (FEM-P1) produced small nodal errors across the investigated mesh resolutions. At 8 degrees of freedom, the maximum nodal error was 1.022×10^{-5}. Increasing the number of degrees of freedom to 16 and 32 reduced the error to 8.079×10^{-7} and 5.700×10^{-8}, respectively.

Further mesh refinement produced maximum nodal errors of 3.789×10^{-9} at 64 degrees of freedom and 2.443×10^{-10} at 128 degrees of freedom. These results demonstrate consistent error reduction for the selected smooth benchmark.

The statistical convergence analysis yielded an observed nodal convergence order of 3.998, with $R^2 = 1.000$. This convergence rate represents the measured nodal behavior of the implemented finite element prototype for the selected problem rather than the general convergence order of piecewise linear finite elements.

C. Spectral Collocation Method

The Chebyshev spectral collocation method demonstrated rapid numerical convergence for the investigated smooth boundary value problem. The numerical approximation closely matched the exact analytical solution and reached near machine-precision accuracy using relatively few degrees of freedom.

The convergence behavior in Figure 7 shows a rapid reduction in numerical error, followed by a numerical accuracy limit set by floating-point precision. The statistical convergence results in Table 6 characterize this behavior as exponential convergence toward machine precision.

These findings demonstrate the high numerical accuracy of spectral collocation for the selected smooth one-dimensional benchmark.

D. Shooting Method

We evaluated the shooting method by determining the unknown initial slope required to satisfy the prescribed boundary conditions. The numerical procedure employed a secant root-finding strategy to obtain the initial slope.

As reported in Table 7, the shooting method recovered an initial slope of 3.14159265, matching the reported exact slope of 3.14159265 to the displayed precision.

The absolute error in the recovered initial slope was 1.499×10^{-12}, and the numerical procedure converged after two iterations. These results demonstrate accurate recovery of the required initial condition for the investigated linear boundary value problem.

E. Comparative Numerical Accuracy and Convergence

The comparative results demonstrate that all four investigated methods successfully approximated the selected boundary value problem, although their reported numerical errors and convergence characteristics differed.

Figure 6 illustrates the close agreement between the exact analytical solution and the finite difference, finite element, and spectral approximations. The numerical solutions reproduce the analytical solution profile across the computational domain.

At 128 degrees of freedom, the finite difference method achieved a maximum error of 4.942×10^{-5}, whereas the finite element prototype produced a maximum nodal error of 2.443×10^{-10}. The spectral collocation method reached near machine-precision accuracy, while the shooting method recovered the required initial slope with an absolute error of 1.499×10^{-12}.

The convergence analysis reported in Table 6 yielded observed orders of 1.998 and 3.998 for the finite difference and finite element methods, respectively, with $R^2 = 1.000$ for both regression estimates.

Figure 7 illustrates the reduction in numerical error under mesh refinement and the distinct convergence patterns of the investigated discretization methods. The results establish that numerical accuracy and convergence behavior vary according to the selected approximation technique within the investigated benchmark.

F. Computational Characteristics

The computational assessment also considered the numerical methods' structural characteristics. The finite difference discretization produced a tridiagonal algebraic system, as illustrated in Figure 3, while the finite element method employed locally supported piecewise linear basis functions, as shown in Figure 4.

Figure 8 presents an illustrative comparison of computational scaling for sparse direct, iterative or multigrid, and dense or spectral solver families as the number of degrees of freedom increases.

The figure illustrates differences in how computational operations grow with problem size. However, these curves represent illustrative scaling characteristics rather than measured execution times or memory consumption from the numerical experiment. Consequently, the present results establish numerical accuracy and convergence behavior but do not provide a quantitative runtime comparison among the investigated methods.

XII. DISCUSSION

The computational findings demonstrate that the numerical performance of boundary value problem solvers depends on the interaction between discretization strategy, solution regularity, boundary conditions, and the mathematical structure of the governing equation. The comparative experiment provides quantitative evidence of the accuracy and convergence characteristics of finite difference, finite element, spectral collocation, and shooting methods when applied to a smooth one-dimensional Dirichlet boundary value problem. Although all investigated approaches reproduce the analytical benchmark accurately, their convergence behavior and numerical characteristics differ substantially. The finite difference method exhibits the expected second-order convergence, with an observed rate of 1.998. This result confirms the consistency of the centered finite-difference approximation for the smooth benchmark investigated. Increasing the number of degrees of freedom from 8 to 128 reduces the maximum numerical error from 1.006×10^{-2} to 4.942×10^{-5}. The tridiagonal structure of the resulting algebraic system also provides a computational advantage by permitting efficient numerical solution. These characteristics support the use of finite difference methods for boundary value problems on regular domains with structured computational grids. However, their implementation becomes more challenging when complex geometries or irregular boundary configurations must be accommodated.

The linear finite element prototype demonstrates substantially smaller nodal errors for the selected benchmark. The maximum reported nodal error decreases from 1.022×10^{-5} at 8 degrees of freedom to 2.443×10^{-10} at 128 degrees of freedom, with an observed convergence order of 3.998. This unusually high nodal convergence rate should be interpreted as a benchmark-specific result rather than the general convergence order of piecewise linear finite element approximations. The result highlights the importance of distinguishing between nodal accuracy and global approximation error when evaluating finite element performance. Beyond this benchmark, the weak formulation and element-based representation provide a flexible mathematical foundation for problems with nonuniform meshes, complex domains, and heterogeneous material properties. The spectral collocation method achieves near machine-precision accuracy using relatively few degrees of freedom. This behavior is consistent with the rapid convergence expected when global polynomial approximations are applied to sufficiently smooth solutions on regular domains. The result demonstrates the effectiveness of spectral approximation for the selected analytical benchmark. Nevertheless, this accuracy should not be generalized to boundary value problems with discontinuities, sharp boundary layers, or irregular geometries, where the smoothness assumptions that support rapid spectral convergence may no longer hold. The shooting method also demonstrates high numerical accuracy, recovering the required initial slope with an absolute error of 1.499×10^{-12} after two iterations. This finding indicates that transforming the boundary value problem into an initial value formulation is effective for the investigated linear and nonstiff model. However, the numerical reliability of shooting methods depends on how sensitive the initial value problem is to the unknown initial conditions. For stiff, nonlinear, or highly sensitive systems, small errors in the initial estimate may propagate during numerical integration and compromise convergence.

The comparison further demonstrates the complementary roles of classical analytical and modern numerical approaches. Analytical methods provide mathematical insight into the structure of boundary value problems and supply exact solutions for numerical validation. Numerical methods extend the range of solvable problems by approximating systems without closed-form solutions. Integrating analytical benchmarks with numerical error measurements and convergence analysis therefore provides a more informative evaluation than relying on solution profiles alone. The practical applicability of each numerical method must also be considered in relation to the governing equation and computational domain. Finite difference methods provide straightforward discretization for structured problems, while finite element and finite volume formulations accommodate broader classes of geometries, boundary conditions, and conservation requirements. Spectral techniques converge rapidly under appropriate regularity assumptions, whereas iterative and preconditioned solvers become increasingly important for large, sparse algebraic systems. These characteristics indicate that method selection should account for both approximation accuracy and the mathematical requirements of the intended application.

Although the present experiment shows strong numerical agreement, the findings are limited to a single smooth one-dimensional boundary value problem with homogeneous Dirichlet conditions. The reported convergence rates and error magnitudes cannot establish equivalent performance for nonlinear equations, multidimensional domains, discontinuous coefficients, or mixed boundary conditions. Moreover, the present experiment does not systematically compare execution time, memory consumption, or solver robustness across problem classes. Extending the numerical evaluation to these conditions would provide a more comprehensive assessment of the computational advantages and limitations of the investigated methods.

XIII. LIMITATIONS AND FUTURE WORK

This paper includes a reproducible numerical experiment, but the experiment is intentionally simple. It uses one smooth one-dimensional problem with homogeneous Dirichlet conditions. Future work could extend the comparison to nonlinear BVPs such as Bratu's problem, stiff reaction-diffusion systems, discontinuous coefficients, two-dimensional Poisson problems on irregular domains, and adaptive mesh refinement. A stronger empirical study could also compare memory consumption, robustness under noise, preconditioner performance, and a posteriori error estimators.

Future versions of the paper could also include one application-driven case study, such as heat transfer in an irregular domain, beam deflection with mixed boundary conditions, fluid flow with no-slip and outflow conditions, or fractional boundary models with memory effects. These extensions would help move the paper from a methodological comparison toward a more specialized applied mathematics contribution.

XIV. CONCLUSION

This study presented a comparative analytical-numerical framework for investigating classical and modern methods for solving boundary value problems. By integrating analytical solution techniques, numerical discretization, exact benchmark validation, and statistical convergence assessment, the study provides a systematic basis for examining the accuracy and applicability of different solution approaches.

The computational experiment showed distinct numerical characteristics across the methods investigated. The finite difference method achieved an observed convergence order of 1.998, while the linear finite element prototype exhibited a benchmark-specific nodal convergence order of 3.998. At 128 degrees of freedom, the maximum errors reached 4.942×10^{-5} and 2.443×10^{-10} for the finite difference and finite element methods, respectively. Spectral collocation attained near machine-precision accuracy, whereas the shooting method recovered the exact initial slope with an absolute error of 1.499×10^{-12} after two iterations. These findings confirm the importance of discretization strategy, solution smoothness, and numerical validation in assessing computational performance.

The study further shows that analytical and numerical methods complement each other. Classical approaches provide theoretical insight and exact reference solutions, while numerical techniques extend the treatment of boundary value problems to mathematical models for which analytical solutions are unavailable. However, the reported numerical findings apply only to the smooth one-dimensional benchmark studied and should not be generalized to more complex problem classes without additional validation.

Future research should extend the comparative framework to nonlinear and multidimensional boundary value problems, irregular geometries, mixed boundary conditions, and discontinuous coefficients. Incorporating systematic runtime, memory-consumption, and stability assessments would further strengthen the evaluation of computational efficiency and numerical reliability. These extensions would support a more comprehensive framework for selecting and validating boundary value problem solvers in applied mathematics and engineering.

Author Contributions

The author conducted the conceptualization, methodology, data analysis, investigation, writing, review, editing, and final approval of the manuscript.

Funding

This research received no external funding.

Data Availability

The dataset will be available from the author upon reasonable request.

Conflicts of Interest

The author declares no conflict of interest.

REFERENCES

[1]. Aayadi, K., Akhlil, K., Ben Aadi, S., et al. (2021). Multivalued nonmonotone dynamic boundary condition. Boundary Value Problems, 2021, 43. https://doi.org/10.1186/s13661-021-01517-6

[2]. Abdella, K., & Ross, G. (2020). Solving integro-differential boundary value problems using Sinc-derivative collocation. Mathematics, 8(9), 1637. https://doi.org/10.3390/math8091637

[3]. Abdella, K., & Trivedi, J. (2020). Solving multi-point boundary value problems using Sinc-derivative interpolation. Mathematics, 8(12), 2104. https://doi.org/10.3390/math8122104

[4]. Abdelwahed, M., & Chorfi, N. (2020). Spectral discretization of time-dependent vorticity-velocity-pressure formulation of the Navier-Stokes equations. Boundary Value Problems, 2020, 152. https://doi.org/10.1186/s13661-020-01448-8

[5]. Abdelwahed, M., & Chorfi, N. (2022). Numerical resolution of the wave equation using the spectral method. Boundary Value Problems, 2022, 15. https://doi.org/10.1186/s13661-022-01601-5

[6]. Abdelwahed, M., Al Salam, A., & Chorfi, N. (2018). Solving the singular two-dimensional fourth order problem by the mortar spectral element method. Boundary Value Problems, 2018, 39. https://doi.org/10.1186/s13661-018-0960-8

[7]. Abdelwahed, M., Chorfi, N., & Ouertani, H. (2020). The spectral implementation of the nonstationary Stokes problem with nonstandard boundary conditions. Boundary Value Problems, 2020, 94. https://doi.org/10.1186/s13661-020-01387-4

[8]. Abdelwahed, M., Chorfi, N., Mezghani, N., et al. (2022). An algorithm for solving the Navier-Stokes problem with mixed boundary conditions. Boundary Value Problems, 2022, 94. https://doi.org/10.1186/s13661-022-01678-y.

[9]. Calamai, A., Marcelli, C., & Papalini, F. (2018). Boundary value problems for singular second order equations. Fixed Point Theory and Applications, 2018, 20. https://doi.org/10.1186/s13663-018-0645-0

[10]. Cao, J., Wang, Z., Cao, W., et al. (2017). A mixed Legendre-Galerkin spectral method for the buckling problem of simply supported Kirchhoff plates. Boundary Value Problems, 2017, 34. https://doi.org/10.1186/s13661-017-0767-z

[11]. Chen, J. (2020). A fast multiscale Galerkin algorithm for solving boundary value problem of the fractional Bagley-Torvik equation. Boundary Value Problems, 2020, 91. https://doi.org/10.1186/s13661-020-01391-8

[12]. Daba, I. T., & Duressa, G. F. (2022). Fitted numerical method for singularly perturbed Burger-Huxley equation. Boundary Value Problems, 2022, 102. https://doi.org/10.1186/s13661-022-01681-3

[13]. He, Z., Ma, R., & Xu, M. (2018). Three positive solutions for second-order periodic boundary value problems with sign-changing weight. Boundary Value Problems, 2018, 93. https://doi.org/10.1186/s13661-018-1011-1

[14]. Huang, Q., Li, L., & Ouyang, Z. (2019). Asymptotic solutions on multiple solutions arising from laminar flow in a uniformly porous channel with expanding or contracting walls. Boundary Value Problems, 2019, 39. https://doi.org/10.1186/s13661-019-1148-6

[15]. Jin, S., & Luo, Z. (2019). A reduced-order extrapolating collocation spectral method based on POD for the 2D Sobolev equations. Boundary Value Problems, 2019, 63. https://doi.org/10.1186/s13661-019-1176-2

[16]. Khaleghi, M., Babolian, E., & Abbasbandy, S. (2017). Chebyshev reproducing kernel method: Application to two-point boundary value problems. Advances in Difference Equations, 2017, 26. https://doi.org/10.1186/s13662-017-1089-2

[17]. Li, J., Huang, P., Su, J., et al. (2019). A linear, stabilized, non-spatial iterative, partitioned time stepping method for the nonlinear Navier-Stokes/Navier-Stokes interaction model. Boundary Value Problems, 2019, 115. https://doi.org/10.1186/s13661-019-1220-2

[18]. Luo, Z., & Gao, J. (2017). Stabilized mixed finite element model for the 2D nonlinear incompressible viscoelastic fluid system. Boundary Value Problems, 2017, 56. https://doi.org/10.1186/s13661-017-0788-7

[19]. Nuchpong, C., Ntouyas, S. K., Vivek, D., et al. (2021). Nonlocal boundary value problems for psi-Hilfer fractional-order Langevin equations. Boundary Value Problems, 2021, 34. https://doi.org/10.1186/s13661-021-01511-y

[20]. Roul, P. (2019). A fast and accurate computational technique for efficient numerical solution of nonlinear singular boundary value problems. International Journal of Computer Mathematics, 96(1), 51–72. https://doi.org/10.1080/00207160.2017.1417588

[21]. Sapagovas, M., Stikoniene, O., Jakubeliene, K., & Ciupaila, R. (2019). Finite difference method for boundary value problem for nonlinear elliptic equation with nonlocal conditions. Boundary Value Problems, 2019, 94. https://doi.org/10.1186/s13661-019-1202-4

[22]. Teng, F., Luo, Z., & Yang, J. (2017). A natural boundary element method for the Sobolev equation in the 2D unbounded domain. Boundary Value Problems, 2017, 179. https://doi.org/10.1186/s13661-017-0910-x

[23]. Wang, S., & Zhou, Z. (2021). Three solutions for a partial discrete Dirichlet boundary value problem with p-Laplacian. Boundary Value Problems, 2021, 39. https://doi.org/10.1186/s13661-021-01514-9

[24]. Xia, H., & Luo, Z. (2017). Stabilized finite volume element method for the 2D nonlinear incompressible viscoelastic flow equation. Boundary Value Problems, 2017, 130. https://doi.org/10.1186/s13661-017-0862-1

[25]. Xu, C., Zhou, J., Shi, D., et al. (2018). Low order nonconforming finite element method for time-dependent nonlinear Schrödinger equation. Boundary Value Problems, 2018, 174. https://doi.org/10.1186/s13661-018-1093-9

[26]. Yao, W., Guo, Z., & Sun, J. (2018). Periodic boundary value problems for two classes of nonlinear fractional differential equations. Boundary Value Problems, 2018, 172. https://doi.org/10.1186/s13661-018-1092-x

[27]. Zhang, Y., Mei, L., & Lin, Y. (2021). A new method for high-order boundary value problems. Boundary Value Problems, 2021, 48. https://doi.org/10.1186/s13661-021-01527-4

[28]. Zhang, Y., Xu, C., & Zhou, J. (2017). Convergence of a linearly extrapolated BDF2 finite element scheme for viscoelastic fluid flow. Boundary Value Problems, 2017, 140. https://doi.org/10.1186/s13661-017-0872-z

[29]. Zheng, Z., Cai, J., Li, K., et al. (2018). A discontinuous Sturm-Liouville problem with boundary conditions rationally dependent on the eigenparameter. Boundary Value Problems, 2018, 103. https://doi.org/10.1186/s13661-018-1023-x