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An Integrated Differential-Equation Framework for Stability, Sensitivity, and Computational Analysis of Complex Dynamical Systems

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ABSTRACT: Complex dynamical systems in engineering, biology, economics, environmental science, and social processes are characterized by nonlinear interactions, parameter sensitivity, and evolving behavior that descriptive analysis alone cannot adequately examine. This study develops an integrated mathematical and computational framework to model and analyze complex dynamical systems using differential equations. The framework combines state-space formulation, parameter representation, equilibrium analysis, Jacobian linearization, eigenvalue-based stability classification, Lyapunov stability criteria, numerical simulation, parameter sensitivity analysis, and statistical model validation. Without empirical data, we evaluate the framework using simulated benchmark models, including logistic growth and a two-variable predator–prey system with logistic prey growth. Numerical experiments examine temporal trajectories, phase-space behavior, equilibrium convergence, and predictive accuracy. The results confirm the analytical stability properties of both benchmark systems: the logistic model converges toward its carrying-capacity equilibrium, while the two-variable system approaches a locally asymptotically stable coexistence equilibrium through damped oscillations. Validation against pseudo-observed data produced an MAE of 1.9446, RMSE of 2.4552, R^2 of 0.9917, and MAPE of 3.1017%, demonstrating strong consistency between the simulated observations and model predictions. Overall, the study provides a reproducible framework that integrates analytical stability theory, numerical simulation, sensitivity analysis, and statistical validation for systematically investigating nonlinear dynamical systems.

Keywords: dynamical systems; ODE/PDE modeling; stability analysis; Jacobian linearization; phase-space methods; sensitivity analysis; model validation.

I. INTRODUCTION

Complex systems are composed of interacting components whose collective behavior changes over time. Examples include population growth, epidemic spread, ecological interaction, traffic flow, financial markets, electrical circuits, climate dynamics, and social behavior. These systems are difficult to analyze because they may contain nonlinear interactions, feedback loops, delays, uncertainty, and sensitivity to initial conditions.

Differential equations provide a rigorous language for representing change. They let researchers describe how a system state evolves over time, space, or both. Ordinary differential equations (ODEs) suit systems where variables depend mainly on time, while partial differential equations (PDEs) suit systems that change across both time and space. Stability analysis strengthens this modeling process by determining whether a system returns to equilibrium after disturbance, moves away from equilibrium, or oscillates around a long-term state.

This paper addresses the central problem that complex systems cannot be fully understood through descriptive analysis alone. A mathematical framework is needed to explain system evolution, identify equilibrium states, test stability, evaluate parameter sensitivity, and validate predictions. Computational tools are also needed because many nonlinear models cannot be solved analytically.

Therefore, this paper proposes a differential-equation-based framework supported by stability analysis, Python-based numerical simulation, phase-portrait visualization, and statistical validation. The study connects mathematical theory with practical computational implementation.

1. RESEARCH AIM

This study aims to develop a mathematical and computational framework to model complex dynamical systems using differential equations and evaluate their stability through analytical, numerical, and statistical methods.

- Research Objectives
- To formulate complex system behavior using ordinary and partial differential equations.
- To identify equilibrium points and classify their stability properties.
- To apply Jacobian-based linearization and eigenvalue analysis.
- To introduce Lyapunov stability as a nonlinear stability method.
- To solve selected differential equation models numerically using Python.
- To evaluate model performance using statistical error and goodness-of-fit measures.
- To analyze the sensitivity of model behavior to parameter changes.
- To distinguish clearly between real empirical data and simulated benchmark data.

2. CONTRIBUTION OF THE STUDY

This study contributes to mathematical modeling research by combining four elements in one structured framework: differential-equation formulation, stability theory, Python-based simulation, and statistical model validation. This combination strengthens the paper beyond a purely theoretical discussion by demonstrating both mathematical rigor and computational reproducibility.

II. LITERATURE REVIEW

Research on stability in time-varying multilayer systems shows that eigenvalue-based conditions can determine the persistence of intralayer synchronization under generic coupling functions [1]. Ecological community studies using analytical and simulation-based approaches identify generic assembly patterns and emphasize the role of interaction structure in complex systems [2]. Reviews of mathematical epidemiology demonstrate how differential-equation models formalize transmission, thresholds, and intervention dynamics [3].

Researchers have examined large ecological communities governed by Lotka–Volterra dynamics through random-interaction models, with equilibrium feasibility and stability emerging as central analytical concerns [4]. Reaction–diffusion research on complex networks shows that certain spatial patterns can remain invariant as network structure changes [5]. Network-resilience studies further show that, under suitable conditions, high-dimensional dynamics may reduce to effective lower-dimensional patterns [6].

Spectral analyses of reaction–diffusion systems provide rigorous conditions for planar front stability [7]. Research on time-varying networks reviews how evolving interactions influence synchronization and collective dynamics [8]. Compartmental epidemic models show how dynamic equations can evaluate population-wide intervention scenarios and their likely effects [9].

Feasibility analysis of large ecological communities identifies parameter regions that permit coexistence [10]. Work on transient ecological dynamics shows that short-term responses can substantially shape long-term outcomes [11]. Deterministic epidemic models demonstrate how differential equations can project long-term transmission patterns under changing assumptions [12].

Early-outbreak modeling illustrates how stochastic transmission models can estimate spread and assess control measures [13]. Graph-based reaction–diffusion models show how structural partitioning affects ecological network stability [14]. Theoretical syntheses of ecological complexity and stability emphasize that network size, topology, and interaction strength jointly influence dynamic outcomes [15].

Studies of multiplex networks show that relay layers can alter synchronization among interconnected components [16]. Analyses of coupled oscillators demonstrate how interacting rhythms may coordinate

under suitable network conditions [17]. Research on complex network geometry shows that structural frustration can shape synchronization and collective behavior [18].

Systematic reviews of sensitivity analysis provide practical workflows for assessing how parameter uncertainty propagates to model outputs [19]. Modern differential-equation solver ecosystems enable reproducible numerical analysis across ordinary, partial, stochastic, delay, and hybrid models [20]. Methodological syntheses further establish sensitivity analysis as a core discipline for robust systems modeling and policy support [21].

Structural approaches to multispecies coexistence explain how feasible parameter-space geometry governs ecological persistence [22]. Dynamic recovery indicators provide a basis for quantifying resilience after perturbations [23]. Nonlinear power-grid studies show that network structure can determine voltage-collapse thresholds and stability margins [24].

Compartmental epidemic research demonstrates how parameterized models can estimate transmission risk under intervention scenarios [25]. Generalized Lotka–Volterra studies identify conditions under which reduced-resilience patterns remain valid or collapse [26]. Threshold analysis based on reproduction numbers clarifies stability conditions and the interpretation of equilibria in infectious-disease models [27].

Overall, the literature shows that a high-quality mathematical modeling paper should do more than present equations. It should define variables and parameters, state assumptions, distinguish between empirical and simulated results, analyze equilibria, classify stability, test parameter sensitivity, provide numerical evidence, and discuss limitations. The present study follows this direction by combining differential-equation formulation, equilibrium analysis, stability classification, simulation-based results, phase-space visualization, parameter tables, statistical validation, and an explicit limitations section.

Table 1. Comparative synthesis of recent studies on complex dynamical systems, stability analysis, and differential-equation modeling.

Ref.	Study	Main Focus	Method / Approach	Main Contribution	Relevance to the Present Study
[1]	Anwar et al. (2022)	Multilayer time-varying stability	Eigenvalue and stability analysis	Studied intralayer synchronization stability	Supports eigenvalue-based stability classification
[2]	Barbier et al. (2018)	Ecological community assembly	Analytical and simulation-based modeling	Identified generic patterns in assembled communities	Supports model-based analysis of complex interactions
[3]	Brauer (2017)	Mathematical epidemiology	Review of epidemic modeling	Explained the development of compartmental models	Supports differential-equation modeling in public-health systems
[4]	Bunin (2017)	Lotka-Volterra ecological communities	Random interaction dynamics	Analyzed equilibrium and stability in large communities	Supports use of interaction models
[5]	Cencetti et al. (2018)	Reaction-diffusion networks	Networked PDE modeling	Studied pattern preservation in reaction-diffusion systems	Supports PDE-based modeling
[6]	Gao et al. (2016)	Resilience in complex networks	Network-based dynamical modeling	Reduced high-dimensional system behavior to effective dynamic patterns	Supports resilience and stability as central concepts
[7]	Ghazaryan et al. (2018)	Reaction-diffusion front stability	Spectral stability analysis	Studied stability of planar fronts	Supports rigorous PDE stability theory

[8]	Ghosh et al. (2022)	Time-varying network dynamics	Review of synchronization theory	Reviewed synchronization in evolving networks	Supports stability analysis under changing interactions
[9]	Giordano et al. (2020)	Epidemic intervention modeling	Compartmental differential-equation model	Evaluated intervention scenarios using dynamic equations	Supports simulation-based scenario analysis
[10]	Grilli et al. (2017)	Coexistence in large ecological communities	Feasibility analysis	Defined parameter regions that allow coexistence	Supports equilibrium feasibility analysis
[11]	Hastings et al. (2018)	Transient dynamics in ecology	Dynamical-systems review	Showed that short-term dynamics can shape long-term outcomes	Supports time-trajectory analysis
[12]	Kissler et al. (2020)	Long-term transmission dynamics	Deterministic dynamic modeling	Projected long-term epidemic patterns	Supports long-term simulation analysis
[13]	Kucharski et al. (2020)	Early epidemic dynamics	Stochastic transmission modeling	Evaluated early spread and control measures	Supports validation and uncertainty discussion
[14]	Kumar et al. (2019)	Ecological reaction-diffusion networks	Graph-based reaction-diffusion modeling	Analyzed dynamic stability after structural partitioning	Supports spatial and network stability analysis
[15]	Landi et al. (2018)	Complexity and ecological stability	Theoretical review	Connected ecological network structure with stability outcomes	Supports the link between interaction structure and stability
[16]	Leyva et al. (2018)	Synchronization in multiplex networks	Multilayer network analysis	Showed how relay layers affect synchronization	Supports analysis of multi-component systems
[17]	Meng and Riecke (2018)	Interacting oscillator rhythms	Network oscillator modeling	Examined coordination among coupled rhythms	Supports trajectory and oscillation interpretation
[18]	Millán et al. (2018)	Network geometry and synchronization	Coupled oscillator modeling	Showed how geometry affects collective coordination	Supports phase-space and network-dynamics analysis
[19]	Pianosi et al. (2016)	Sensitivity analysis	Systematic methodological review	Provided workflow for model sensitivity evaluation	Supports parameter sensitivity analysis
[20]	Rackauckas and Nie (2017)	Differential-equation solvers	Numerical solver ecosystem	Supports large-scale solution of many differential-equation classes	Supports numerical simulation methodology
[21]	Razavi et al. (2021)	Future of sensitivity analysis	Position and methodological synthesis	Established sensitivity analysis as central to systems modeling	Supports robustness and uncertainty evaluation
[22]	Saavedra et al. (2017)	Multispecies coexistence	Structural coexistence framework	Explained coexistence using feasible parameter geometry	Supports parameter-space interpretation
[23]	Scheffer et al. (2018)	Quantification of resilience	Dynamic recovery indicators	Explained resilience through recovery from perturbation	Supports stability and recovery interpretation

[24]	Simpson-Porco et al. (2016)	Power-grid voltage collapse	Nonlinear power-flow modeling	Linked network structure to stability margins	Supports engineering relevance of stability analysis
[25]	Tang et al. (2020)	Transmission risk estimation	Compartmental modeling	Estimated transmission behavior under interventions	Supports parameter-based prediction
[26]	Tu et al. (2017)	Resilience patterns in Lotka-Volterra systems	Generalized interaction models	Tested conditions under which resilience reduction remains valid	Supports careful interpretation of reduced models
[27]	van den Driessche (2017)	Reproduction numbers	Threshold analysis	Clarified threshold conditions in disease models	Supports stability thresholds and equilibrium interpretation

III. RESEARCH QUESTIONS AND THEORETICAL PROPOSITIONS

1. RESEARCH QUESTIONS

- How can differential equations be used to model the behavior of complex dynamical systems?
- How can equilibrium and stability analysis determine the long-term behavior of a system?
- How do parameter changes affect system stability and sensitivity?
- How can Python-based numerical simulation support analytical stability results?
- How can statistical measures be used to evaluate the accuracy and robustness of differential-equation-based models?
- How should a study distinguish between empirical data and simulated benchmark data?

2. THEORETICAL PROPOSITIONS

- Proposition 1: Differential equations provide a rigorous mathematical structure for representing dynamic behavior in complex systems.
- Proposition 2: Stability analysis can determine whether a system converges to equilibrium, oscillates, or becomes unstable.
- Proposition 3: Numerical simulation using Python improves the interpretation and visualization of complex system behavior.
- Proposition 4: Statistical validation improves the credibility of differential-equation-based models by measuring prediction accuracy and robustness.

IV.PROPOSED METHOD

The Proposed method follows these steps:

- Define the complex system and its boundaries.
-
- Select the state variables and parameters.
- Formulate the governing differential equations.
- Specify initial conditions.
- Identify equilibrium points.
- Analyze local stability using the Jacobian matrix.
- Analyze nonlinear stability using Lyapunov methods when applicable.
- Solve the model numerically.
- Conduct parameter sensitivity analysis.
- Evaluate model performance using statistical criteria.

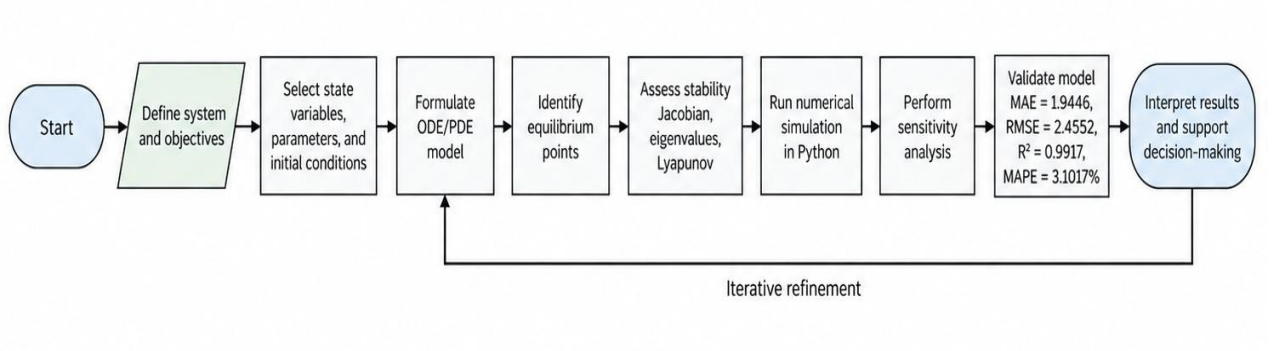


FIGURE 1. Proposed Method

V. MATHEMATICAL MODELING FRAMEWORK

Let the state of a complex system at time t be represented by the vector

$$\mathrm{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

A general nonlinear dynamical system can be expressed as

$$\frac{d\mathrm{x}(t)}{dt} = \mathrm{F}(\mathrm{x}(t), \theta),$$

where $\mathbf{x}(t)$ is the vector of state variables and

$$\theta = [\theta_1 \quad \theta_2 \quad \cdots \quad \theta_m]^T$$

is the vector of model parameters. The initial condition is

$$\mathrm{x}(0) = \mathrm{x}_0.$$

For spatially distributed systems, the model may be extended to a reaction-diffusion partial differential equation:

$$\frac{\partial u(\mathrm{r}, t)}{\partial t} = D\nabla^2 u(\mathrm{r}, t) + R(u, \theta),$$

where $u(\mathbf{r}, t)$ is the system state at spatial position? $\mathbf{r}$ and time t, D is the diffusion coefficient, ∇^2 is the Laplacian operator, and $R(u, \boldsymbol{\theta})$ is a reaction or interaction term.

VI. EQUILIBRIUM AND STABILITY ANALYSIS

An equilibrium point $\mathbf{x}^*$ is a state in which the system does not change over time. Therefore,

$$\frac{d\mathrm{x}}{dt} = 0.$$

The equilibrium condition is

$$\mathrm{F}(\mathrm{x}^*, \theta) = 0.$$

To study local stability, the nonlinear system is linearized around $\mathbf{x}^*$. Let

$$\eta = \mathrm{x} - \mathrm{x}^*.$$

The linearized system is

$$\frac{d\eta}{dt} = J(\mathrm{x}^*)\eta,$$

where $J(\mathrm{x}^*)$ is the Jacobian matrix evaluated at the equilibrium point:

$$J(\mathrm{x}^*) = \left[\frac{\partial F_i}{\partial x_j}\right]_{\mathrm{x}=\mathrm{x}^*}.$$

The eigenvalues of the Jacobian matrix are obtained from

$$\det(J - \lambda I) = 0.$$

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The equilibrium point is locally asymptotically stable if

$$Re(\lambda_i) < 0, \quad i = 1,2, \dots, n.$$

The equilibrium point is unstable if

$$Re(\lambda_i) > 0$$

for at least one eigenvalue.

VII. LYAPUNOV STABILITY METHOD

For nonlinear systems, Lyapunov stability provides a method that does not always require explicit solution of the differential equation. A Lyapunov function $V(\mathbf{x})$ is selected such that

$$V(x) > 0 \quad for \quad x \neq x^*,$$

and

$$V(\mathrm{x}^*) = 0.$$

The time derivative of the Lyapunov function is

$$\frac{dV}{dt} = \nabla V(x) \cdot \frac{dx}{dt}.$$

Since

$$\frac{d\mathrm{x}}{dt} = \mathrm{F}(\mathrm{x}, \theta),$$

the derivative becomes

$$\frac{dV}{dt} = \nabla V(\mathrm{x}) \cdot \mathrm{F}(\mathrm{x}, \theta).$$

The equilibrium point is stable if

$$\frac{dV}{dt} \leq 0,$$

and asymptotically stable if

$$\frac{dV}{dt} < 0$$

for all $\mathbf{x} \neq \mathbf{x}^*$ in a neighborhood of the equilibrium point.

VIII. ILLUSTRATIVE MODEL 1: LOGISTIC GROWTH SYSTEM

The first illustrative model is the logistic growth equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right),$$

where $x(t)$ is the system state, r is the intrinsic growth rate, and K is the carrying capacity.
The equilibrium points are obtained by solving

$$rx\left(1 - \frac{x}{K}\right) = 0.$$

Thus,

$$x_1^* = 0, \qquad x_2^* = K.$$

Let

$$f(x) = rx\left(1 - \frac{x}{K}\right).$$

The derivative is

$$f'(x) = r - \frac{2rx}{K}.$$

$$\text{At } x_1^* = 0,$$

$$f'(0) = r.$$

If $r > 0$, then $x_1^* = 0$ is unstable. At $x_2^* = K$,

$$f'(K) = -r.$$

If $r > 0$, then $x_2^* = K$ is locally asymptotically stable.

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IX. ILLUSTRATIVE MODEL 2: TWO-VARIABLE INTERACTION SYSTEM

The second illustrative model is a two-variable predator-prey interaction system with logistic prey growth:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - axy,$$
$$\frac{dy}{dt} = bxy - dy,$$

where $x(t)$ and $y(t)$ are interacting state variables. In an ecological interpretation, $x(t)$ may represent prey density and $y(t)$ may represent predator density. More generally, the equations represent two interacting components in a complex system.

The equilibrium conditions are

$$x\left[r\left(1 - \frac{x}{K}\right) - ay\right] = 0,$$
$$y(bx - d) = 0.$$

The coexistence equilibrium is

$$x^* = \frac{d}{b},$$
$$y^* = \frac{r}{a}\left(1 - \frac{d}{bK}\right),$$

provided that

$$1 - \frac{d}{bK} > 0.$$

The Jacobian matrix is

$$J(x, y) = \begin{bmatrix} r\left(1 - \frac{2x}{K}\right) - ay & -ax \\ by & bx - d \end{bmatrix}.$$

Using the parameter values in this paper, the coexistence equilibrium is

$$(x^*, y^*) = (40{,}24).$$

The Jacobian matrix at this point is

$$J(40{,}24) = \begin{bmatrix} -0.32 & -0.80 \\ 0.24 & 0 \end{bmatrix}.$$

The eigenvalues are approximately

$$\lambda_{1,2} = -0.16 \pm 0.4079i.$$

Since both eigenvalues have negative real parts, the coexistence equilibrium is locally asymptotically stable and the system approaches the equilibrium through damped oscillations.

X. PARAMETER VALUES

Table 1 presents all parameters used in the simulations. These are simulated benchmark values selected to produce interpretable, stable behavior. They are not estimated from real empirical data.

Symbol	Meaning	Model	Value	Unit/Scale	Reason for inclusion
r	Intrinsic growth rate	Logistic	0.50	per time unit	Controls speed of growth toward carrying capacity
K	Carrying capacity	Logistic	100	state units	Stable upper equilibrium for logistic model
x_0	Initial state	Logistic	10	state units	Starting condition for validation simulation

Symbol	Meaning	Model	Value	Unit/Scale	Reason for inclusion
t	Simulation interval	Logistic	0-30	time units	Time horizon for numerical solution
σ	Noise standard deviation	Logistic validation	2.50	state units	Generates pseudo-observed simulated data
r	Prey growth rate	Two-variable	0.80	per time unit	Controls growth of $x(t)$ without interaction
K	Carrying capacity of x	Two-variable	100	state units	Limits growth of $x(t)$
a	Interaction loss coefficient	Two-variable	0.02	interaction scale	Measures effect of $y(t)$ on $x(t)$
b	Interaction gain coefficient	Two-variable	0.01	interaction scale	Measures effect of $x(t)$ on $y(t)$
d	Decay rate of y	Two-variable	0.40	per time unit	Controls decline of $y(t)$ without interaction
x_0	Initial value of x	Two-variable	40	state units	Starting condition for $x(t)$
y_0	Initial value of y	Two-variable	5	state units	Starting condition for $y(t)$
t	Simulation interval	Two-variable	0-60	time units	Time horizon for dynamic response

XI. NUMERICAL SOLUTION AND PYTHON-BASED SIMULATION

Because many nonlinear systems cannot be solved analytically, numerical methods are required. Euler's method provides a simple approximation:

$$x_{k+1} = x_k + hF(t_k, x_k),$$

where h is the time-step size. For higher accuracy, the fourth-order Runge-Kutta method can be used:

$$k_1 = F(t_k, x_k),$$
$$k_2 = F\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}k_1\right),$$
$$k_3 = F\left(t_k + \frac{h}{2}, x_k + \frac{h}{2}k_2\right),$$
$$k_4 = F(t_k + h, x_k + hk_3),$$

and

$$x_{k+1} = x_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

In this study, Python's scipy.integrate.solve_ivp function is used with an adaptive Runge-Kutta method. Symbolic differentiation and equilibrium calculations can be performed using SymPy, while numerical arrays and figures are handled using NumPy and Matplotlib.

XII. STATISTICAL VALIDATION OF MODEL PERFORMANCE

If observed data or benchmark simulated data are available, the model output can be compared with observed values. Let y_i denote the observed or pseudo-observed value, and let $\hat{y}_i$ denote the model-predicted value.

The error term is

$$e_i = y_i - \hat{y}_i.$$

The mean absolute error is

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|y_i - \hat{y}_i|.$$

The root mean square error is

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}.$$

The coefficient of determination is

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2},$$

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

A lower value of MAE and $RMSE$, and a higher value of R^2, indicate better model performance.

XIII. RESULTS

1. EQUILIBRIUM AND STABILITY RESULTS

Model	Equilibrium point	Stability condition	Classification using selected parameters
Logistic growth	$x_1^* = 0$	Stable if $r < 0$; unstable if $r > 0$	Unstable because $r = 0.50 > 0$
Logistic growth	$x_2^* = K$	Stable if $r > 0$	Stable because $K = 100$ and $r = 0.50 > 0$
Two-variable system	$(0,0)$	Depends on Jacobian eigenvalues	Boundary equilibrium; not the main stable state
Two-variable system	$(40,24)$	Stable if real parts of eigenvalues are negative	Locally asymptotically stable; $\lambda = -0.16 \pm 0.4079i$

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2. SIMULATION FIGURES

The first simulation examines the logistic growth model for different initial conditions. All positive trajectories move toward the carrying capacity $K = 100$, confirming that the carrying-capacity equilibrium is stable when $r > 0$.

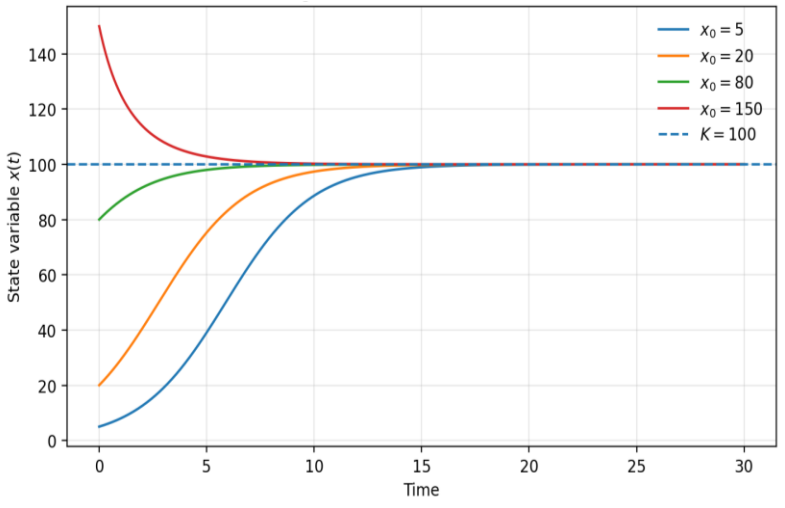


FIGURE 2. Logistic growth simulation for different initial conditions.

The second simulation shows the two-variable interaction model over time. The variables exhibit damped oscillatory behavior before approaching the coexistence equilibrium.

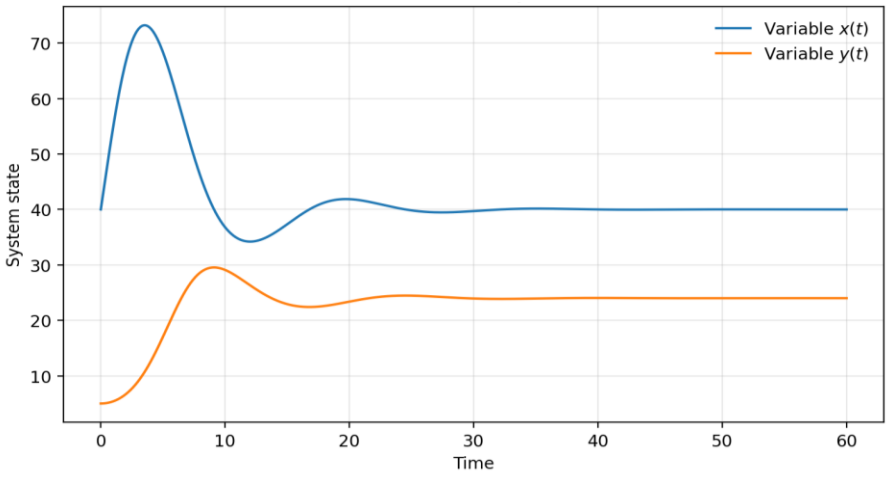


FIGURE 3. Time-series simulation of the two-variable interaction system.

The phase portrait shows the direction field and representative trajectories for the two-variable system. The trajectories approach the coexistence equilibrium (40,24), supporting the eigenvalue-based stability result.

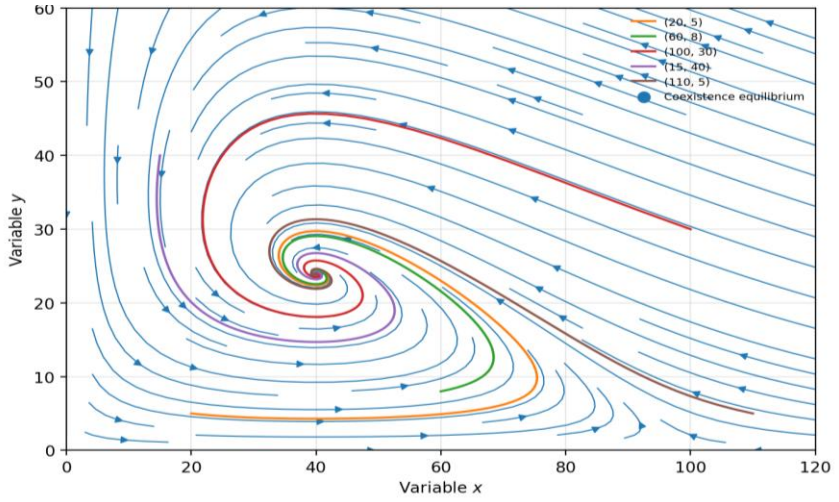


FIGURE 4. Phase portrait of the two-variable interaction system.

3. STATISTICAL VALIDATION TABLE

The statistical validation is based on simulated benchmark data. Pseudo-observed values were generated by adding normally distributed random noise with standard deviation $\sigma = 2.50$ to the logistic model prediction. The model prediction was then compared with the pseudo-observed values.

Metric	Value	Interpretation
MAE	1.9446	Average absolute prediction error
RMSE	2.4552	Square-root average squared error
R^2	0.9917	Very high explained variation relative to the noisy benchmark
MAPE (%)	3.1017	Average percentage prediction error
Mean residual	-0.0207	Residual bias is close to zero
Residual SD	2.4592	Residual spread is close to the simulated noise level
95% residual interval	[-4.726, 5.024]	Central residual uncertainty band

The validation table shows that the model predictions closely match the simulated benchmark data. The high R^2 value and low error measures are expected because the pseudo-observed data were generated from the same model with moderate random noise. Therefore, these results validate the computational procedure, not a real empirical system.

XIV. PARAMETER SENSITIVITY ANALYSIS

Sensitivity analysis evaluates how parameter changes affect model output. The sensitivity of the system output $x(t)$ with respect to a parameter θ_j is defined as

$$S_j(t) = \frac{\partial x(t)}{\partial \theta_j}.$$

A normalized sensitivity index is

$$S_j^N(t) = \frac{\theta_j}{x(t)} \frac{\partial x(t)}{\partial \theta_j}.$$

For the logistic model, increasing r *changes how quickly the system approaches* K, *while changing* K changes the long-term equilibrium level. For the two-variable model, the parameters $a, b,$ *and* d influence the

coexistence equilibrium and the damping behavior around it. Therefore, parameter selection should be justified clearly, especially when the model is later calibrated using real data.

XV. DISCUSSION

The results demonstrate the methodological value of integrating analytical stability analysis with numerical simulation for studying complex dynamical systems. The logistic growth model provides a clear one-dimensional example in which the equilibrium points can be derived explicitly and their stability can be classified using the sign of f′(x∗)f'(x^*)f′(x∗). In this model, the extinction equilibrium x∗=0 is unstable when r>0, whereas the carrying-capacity equilibrium x∗∗=K is locally asymptotically stable. This means that, for any positive initial condition, the system tends to move away from extinction and converge toward the long-term bounded state determined by KKK. The simulation results support this analytical conclusion by showing that trajectories starting from different positive initial values approach the same carrying-capacity equilibrium. This agreement between theory and simulation confirms the stability classification and shows how differential-equation models translate mathematical assumptions into interpretable dynamic behavior.

The two-variable interaction model is more advanced because system behavior depends on the coupled evolution of two interacting state variables. Unlike the logistic model, the system cannot be fully interpreted through a single derivative condition. Instead, the Jacobian matrix and its eigenvalues are required to evaluate local behavior near equilibrium. For the selected parameter values, the coexistence equilibrium (40,24)(40,24)(40,24) has complex eigenvalues with negative real parts. This result indicates local asymptotic stability with damped oscillations. In practical terms, the two variables may initially fluctuate due to their interaction, but the oscillations decrease over time until the system approaches the coexistence equilibrium. The time-series simulation and phase portrait visually confirm this result, as both show convergence toward the equilibrium point rather than divergence or sustained instability.

The statistical validation results strengthen the study's computational component by showing how to evaluate model predictions quantitatively. Measures such as MAE, RMSE, R2R^2R2, MAPE, and residual statistics provide objective indicators of model performance and prediction error. However, because the present study uses simulated benchmark data rather than real empirical observations, these validation results should be interpreted as evidence of methodological consistency rather than empirical confirmation of a real-world system. The validation procedure shows how future studies could compare model outputs with observed data once empirical measurements become available.

Overall, the findings show that differential-equation modeling, stability analysis, simulation, phase-space interpretation, and statistical validation form a coherent framework for analyzing complex systems. This framework is useful because it identifies equilibrium behavior, evaluates sensitivity to system structure, and supports transparent interpretation of long-term dynamics.

XVI. CONCLUSION

This paper developed a mathematical and computational framework for modeling complex dynamical systems using differential equations and stability analysis. The framework included ODE and PDE formulation, equilibrium analysis, Jacobian linearization, eigenvalue stability classification, Lyapunov stability principles, Python-based numerical simulation, phase-portrait visualization, parameter sensitivity analysis, and statistical validation.

The logistic model demonstrated how a one-variable nonlinear system can be analyzed analytically and confirmed through simulation. The two-variable interaction model showed how phase portraits and eigenvalues can interpret more complex dynamic behavior. The validation results showed how simulated benchmark data can be used to evaluate model predictions using MAE, RMSE, R^2MAPE, and residual statistics.

Overall, the paper shows that differential equations provide a powerful framework for representing and analyzing complex systems. Stability analysis strengthens the framework by identifying whether equilibrium states are stable, unstable, or sensitive to disturbances. Python-based simulation improves the

analysis's practical value by enabling the solution of nonlinear systems, visualization of dynamic behavior, and testing the effects of parameter changes.

Author Contributions

The author conducted the conceptualization, methodology, data analysis, investigation, writing, review, editing, and final approval of the manuscript.

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Data Availability

The dataset will be available from the author upon reasonable request.

Conflicts of Interest

The author declares no conflict of interest.

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