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A Generalized Iterative Framework for Solving Nonlinear Integral Equations

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ABSTRACT: Nonlinear integral equations play a fundamental role in modeling complex phenomena across mathematics, physics, engineering, and applied sciences. However, obtaining accurate analytical solutions remains challenging, particularly for equations involving nonlinear operators and diverse integration domains. This study develops a generalized relaxation-based iterative framework for the numerical solution of nonlinear Fredholm and Volterra integral equations of the second kind. The proposed approach extends the classical successive approximation method by incorporating an adjustable relaxation parameter into the iterative process, providing additional flexibility in constructing successive approximations. A unified operator formulation is established to accommodate both Fredholm equations with fixed integration limits and Volterra equations with variable upper limits within a consistent mathematical framework. Theoretical convergence is investigated under bounded-kernel, Lipschitz-continuity, and contraction assumptions, establishing sufficient conditions for convergence to a unique solution. The numerical performance of the framework is assessed through three benchmark problems with known exact solutions, encompassing quadratic and cubic nonlinearities. After ten iterations, the maximum absolute errors for the quadratic Fredholm, quadratic Volterra, and cubic Fredholm problems are (1.9189\times10^{-4}), (5.9045\times10^{-11}), and (2.4028\times10^{-6}), respectively. The corresponding mean absolute errors are (9.5943\times10^{-5}), (2.6195\times10^{-11}), and (1.2014\times10^{-6}). These findings demonstrate high numerical accuracy across the investigated benchmark problems, with particularly small approximation errors observed for the quadratic Volterra equation. The numerical results are consistent with the theoretical convergence analysis and demonstrate the applicability of the unified formulation to different nonlinear integral equation structures. The proposed framework offers a mathematically justified and straightforward computational approach for approximating solutions to nonlinear Fredholm and Volterra integral equations under suitable contraction conditions.

Keywords: nonlinear integral equations; Fredholm integral equations; Volterra integral equations; iterative methods; relaxation parameter; convergence analysis; numerical approximation.

I. INTRODUCTION

Integral equations are important mathematical models in applied mathematics and many scientific fields. They appear in physical systems, population models, potential theory, heat transfer, fluid mechanics, electromagnetic theory, and engineering applications. In many models, the unknown function appears inside an integral, making them different from ordinary algebraic or differential equations.

Nonlinear integral equations are more challenging than linear integral equations because the unknown function appears through a nonlinear expression such as $u(t)^2$, $u(t)^3$, or a more general nonlinear function $F(u(t))$. In most practical cases, obtaining exact closed-form solutions is difficult. For this reason, researchers widely use numerical and iterative methods to construct approximate solutions.

The present study focuses on nonlinear Fredholm and Volterra integral equations of the second kind. The Fredholm case has a fixed integration interval, while the Volterra case has an upper limit that depends on the independent variable. Treating both types in the same study strengthens the generality of the proposed approach and shows that the method is not limited to a single class of integral equations.

The main idea of the proposed method is to construct a sequence of approximate solutions that converges to the exact solution under suitable assumptions. We introduce a relaxation parameter to generalize the classical successive approximation method. When the relaxation parameter equals one, the proposed method reduces to the classical iteration. When the parameter lies between zero and one, it moderates the update and improves stability in some nonlinear problems.

The objectives of this research are: (i) to formulate a unified iterative framework for nonlinear Fredholm and Volterra integral equations, (ii) to provide a convergence analysis based on contraction conditions, and (iii) to validate the method using numerical examples with known exact solutions.

II. LITERATURE REVIEW

Recent research on nonlinear integral equations has combined linearization, basis approximation, and collocation to improve numerical accuracy. Researchers have applied Newton-Kantorovich iteration with Haar wavelets to nonlinear Fredholm and Volterra equations, used radial basis functions for nonlinear Volterra equations, and developed sigmoid collocation for nonlinear Fredholm problems [1], [2], and [3]. Although these methods can produce accurate approximations, their performance depends on wavelet resolution, basis selection, or collocation design.

Polynomial and wavelet-based approximations have also been used for multidimensional and mixed integral equations. Bernstein polynomials provide a convergent approximation for general two-dimensional nonlinear Fredholm kernels, polynomial quasi-wavelets address mixed Volterra-Fredholm equations in the complex plane, and modified moving least-squares approximation extends the analysis to three-dimensional mixed equations [4], [5], and [6]. These methods broaden the applicable problem class but may lead to large algebraic systems and increased computational cost.

Other studies have emphasized convergence domains, reproducing-kernel constructions, and sparse spectral representations. Researchers have examined global convergence conditions for Newton-type methods for nonlinear integral operators, used reproducing-kernel spaces for two-dimensional nonlinear Volterra equations, and proposed sparse spectral techniques for nonlinear Volterra integro-differential equations with general kernels [7], [8], and [9]. Their theoretical strengths are important, although implementation can be more specialized than a direct fixed-point iteration.

Mesh-free and Newton-assisted procedures provide further alternatives. Researchers have applied radial basis-function approximation to two-dimensional nonlinear Volterra-Fredholm equations with error analysis, and used an integral mean-value formulation combined with Newton iteration for nonlinear Fredholm equations. Sinc-Nystrom discretization has likewise been developed for nonlinear Volterra equations of the first kind [10], [11], and [12]. These studies report high accuracy, but each method is tailored to a specific equation form.

Researchers have extended Sinc, Chebyshev, and Bernstein representations to higher-dimensional problems. Researchers have used a Sinc-Nystrom scheme for two-dimensional nonlinear Fredholm equations, proposed an adaptive Chebyshev operational-vector method for nonlinear Volterra equations, and applied Bernstein approximation to three-dimensional Volterra-Fredholm equations [13], [14], and [15]. The resulting reductions are computationally useful, although the dimension and basis order can substantially enlarge the numerical system.

Hybrid collocation and spectral strategies have focused on nonlinear mixed and weakly singular equations. Researchers have used hybrid functions for two-dimensional nonlinear Volterra-Fredholm equations, discrete Legendre spectral methods for weakly singular Hammerstein-Fredholm equations, and Picard iteration combined with cubature for mixed Volterra-Fredholm equations [16], [17], and [18]. These

contributions highlight the value of iterative correction, but also indicate the need for acceleration and simpler parameter control.

Successive approximation remains especially relevant because of its direct operator form. Researchers have formulated fixed-point iteration with product integration for weakly singular nonlinear Volterra equations, applied successive approximation with cubature to two-dimensional Fredholm-Volterra equations, and used collocation for mixed Volterra-Fredholm equations of Hammerstein type [19], [20], and [21]. These methods support the feasibility of a unified iteration while showing that convergence speed and discretization accuracy require careful treatment.

Several specialized numerical schemes have also been reported. Researchers have tested a general numerical procedure on nonlinear Fredholm and Volterra equations, used modified Adomian decomposition with quadrature for phase-lag Volterra-Fredholm models, and developed Nystrom-quasilinearization with smoothing for weakly singular nonlinear Fredholm equations [22], [23], and [24]. Their results are promising, but broader benchmark comparisons are needed to establish relative efficiency.

Iterative algorithms and block-pulse representations further demonstrate the diversity of available techniques. A closed-form iterative algorithm has been proposed for a Hammerstein nonlinear equation, whereas block-pulse functions have been used for nonlinear Volterra-Fredholm-Hammerstein equations in two-dimensional spaces [25] and [26]. Collectively, the literature shows that many accurate methods remain equation-specific or require nontrivial algebraic transformations, which motivates a simple relaxed framework applicable to both Fredholm and Volterra equations.

Table 1. Comparative summary of numerical and iterative studies on nonlinear integral equations**.**

No.	Author(s)	Year	Type of integral equation	Method used	Research limitation / future improvement
[1]	Abdul Sathar et al.	2020	Nonlinear Fredholm and Volterra integral equations	Newton-Kantorovich and Haar wavelets	Increasing wavelet resolution may improve accuracy but also increases computational cost.
[2]	Assari & Dehghan	2018	Nonlinear Volterra integral equation	Radial basis function collocation	Computational performance depends on basis selection and collocation points.
[3]	Azevedo	2022	Nonlinear Fredholm integral equation	Sigmoid collocation method	Mainly limited to Fredholm equations and can be extended to Volterra or mixed integral equations.
[4]	Babaaghaie & Maleknejad	2018	Two-dimensional nonlinear Fredholm integral equation	Bernstein polynomial method	Suitable for Fredholm equations but needs broader comparison with iterative acceleration techniques.
[5]	Beiglo & Gachpazan	2020	Nonlinear mixed Volterra-Fredholm integral equation	Polynomial quasi-wavelet method	Developed in the complex plane and may require adaptation for general real-domain problems.
[6]	El Majouti et al.	2022	Three-dimensional nonlinear mixed Volterra-Fredholm integral equation	Modified moving least-squares method	Useful for high-dimensional problems but may require high computational resources.
[7]	Ezquerro & Hernandez-Veron	2019	Nonlinear Fredholm integral equation	Newton's method	Mainly addresses Fredholm equations and leaves room for more general iterative frameworks.

No.	Author(s)	Year	Type of integral equation	Method used	Research limitation / future improvement
[8]	Fazli et al.	2017	Two-dimensional nonlinear Volterra integral equation	Reproducing kernel method	Restricted to two-dimensional Volterra equations and requires extension to mixed equation forms.
[9]	Gutleb	2021	Nonlinear Volterra integro-differential equation	Sparse spectral method	Focuses on integro-differential equations rather than pure nonlinear integral equations.
[10]	Laeli Dastjerdi & Nili Ahmadabadi	2017	Two-dimensional nonlinear Volterra-Fredholm integral equation	Radial basis function approximation	Accuracy depends on the choice of radial basis functions and related parameters.
[11]	Li & Huang	2016	Nonlinear Fredholm integral equation	Integral mean value theorem and Newton iteration	Mainly limited to Fredholm equations; can be extended to mixed Fredholm-Volterra problems.
[12]	Ma et al.	2016	Nonlinear Volterra integral equation	Sinc-Nystrom method	Focuses on Volterra equations of the first kind; further testing is needed for second-kind nonlinear equations.
[13]	Ma et al.	2018	Two-dimensional nonlinear Fredholm integral equation	Sinc-Nystrom method	Focuses only on Fredholm equations and does not provide a unified treatment for Volterra equations.
[14]	Maleknejad & Dehbozorgi	2018	Nonlinear Volterra integral equation	Chebyshev operational vector method	Efficient for Volterra equations but may need modification for Fredholm-Volterra systems.
[15]	Maleknejad et al.	2018	Three-dimensional Volterra-Fredholm integral equation	Bernstein approximation	Handles higher dimensions but may be computationally expensive.
[16]	Maleknejad & Saeedipoor	2020	Two-dimensional nonlinear Volterra-Fredholm integral equation	Hybrid functions and collocation	Reduces the problem to nonlinear algebraic equations, which may increase computational effort.
[17]	Mandal et al.	2021	Weakly singular nonlinear Fredholm-Hammerstein integral equation	Discrete Legendre spectral method	Specialized for weakly singular Hammerstein equations and may be extended to other nonlinear forms.
[18]	Micula	2019	Mixed Volterra-Fredholm integral equation	Picard iteration and cubature formula	Classical iteration may converge slowly, so acceleration or correction strategies can improve performance.
[19]	Micula	2020	Weakly singular nonlinear Volterra integral equation	Fixed-point iteration and product integration	Focuses on Volterra equations and may be extended to Fredholm-Volterra cases.
[20]	Micula	2021	Two-dimensional Fredholm-Volterra integral equation	Successive approximation and cubature formula	Can be improved by adding acceleration parameters or error-control techniques.

No.	Author(s)	Year	Type of integral equation	Method used	Research limitation / future improvement
[21]	Micula	2022	Mixed Volterra-Fredholm Hammerstein integral equation	Collocation method	Collocation-based and comparable to generalized iterative schemes.
[22]	Mohammad et al.	2021	Nonlinear Fredholm and Volterra integral equations	Novel numerical method	Needs further validation using more examples and comparison with classical iterative methods.
[23]	Mosa et al.	2021	Nonlinear Volterra-Fredholm integral equation with phase lag	Modified Adomian decomposition method and quadrature rules	The phase-lag model is specialized, so wider testing on general nonlinear equations is needed.
[24]	Najafi	2020	Weakly singular nonlinear Fredholm integral equation	Nystrom-quasilinearization method	Effective for weakly singular kernels but has a more complex implementation.
[25]	Rabbani	2019	Hammerstein nonlinear integral equation	Iterative algorithm	Designed for a specific Hammerstein form and requires testing on broader nonlinear integral equations.
[26]	Xie et al.	2017	Nonlinear Volterra-Fredholm-Hammerstein integral equation	Block-pulse functions	The algebraic system may become large for complex kernels or higher-dimensional problems.

III. PRELIMINARIES AND PROBLEM FORMULATION

This section presents the basic forms of the nonlinear integral equations considered in this research and introduces the assumptions used in the convergence analysis.

1. NONLINEAR FREDHOLM INTEGRAL EQUATION

A nonlinear Fredholm integral equation of the second kind can be written as

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F(u(t))\,dt, \qquad a \le x \le b. \quad (1)$$

Here, u(x) is the unknown function, f(x) is a known continuous function, λ is a real parameter, K(x,t) is the kernel, and F is a nonlinear function of the unknown solution.

2. NONLINEAR VOLTERRA INTEGRAL EQUATION

A nonlinear Volterra integral equation of the second kind can be written as

$$u(x) = f(x) + \lambda \int_a^x K(x,t)F(u(t))\,dt, \qquad a \le x \le b. \quad (2)$$

The main difference from the Fredholm equation is that the upper integration limit is x. This makes the Volterra equation causal in the sense that the value of u(x) depends only on earlier values of the solution over [a,x].

3. UNIFIED FORM

Both equations may be written in the unified form

$$u(x) = f(x) + \lambda \int_a^{c(x)} K(x,t)F(u(t))\,dt, \qquad a \le x \le b. \quad (3)$$

where c(x) = b gives the Fredholm case and c(x) = x gives the Volterra case. This unified representation allows one iterative framework to be used for both types of equations.

4. ASSUMPTIONS

- The function f is continuous on [a,b].
- The kernel K(x,t) is continuous and bounded on the required domain.
- The nonlinear function F is Lipschitz continuous on a bounded interval containing the approximate solutions.
- The integral operator satisfies a contraction condition under the chosen parameter values.

IV. PROPOSED GENERALIZED ITERATIVE METHOD

Let T be the nonlinear integral operator defined by

$$(Tu)(x) = f(x) + \lambda \int_a^{c(x)} K(x,t)F(u(t))\,dt. \quad (4)$$

Solving the nonlinear integral equation is equivalent to finding a fixed point of T, that is, a function u satisfying Tu = u.

1. BASIC ITERATIVE SCHEME

The classical iterative process starts from an initial approximation $u_0(x)$. In this study, the natural choice is

$$u_0(x) = f(x). \quad (5)$$

The successive approximations are then generated by

$$u_{n+1}(x) = f(x) + \lambda \int_a^{c(x)} K(x,t)F(u_n(t))\,dt, \qquad n = 0,1,2,\ldots. \quad (6)$$

2. RELAXED GENERALIZED SCHEME

To obtain a more flexible framework, the update may be relaxed as follows:

$$u_{n+1}(x) = (1-\omega)u_n(x) + \omega[\, f(x) + \lambda \int_a^{c(x)} K(x,t)F(u_n(t))\,dt\,]. \quad (7)$$

$$0 < \omega \le 1. \quad (8)$$

When ω = 1, the method becomes the classical successive approximation method. When 0 < ω < 1, the new approximation is a weighted average of the previous approximation and the updated integral approximation.

3. COMPUTATIONAL ALGORITHM

- Determine the equation type by setting:

$$c(x) = bc(x) = bc(x) = b, or\ Volterra\ with\ c(x) = xc(x) = xc(x) = x.$$

- Define the initial approximation as:

$$u0(x) = f(x)u_0(x) = f(x)u0(x) = f(x).$$

- Choose the relaxation parameter

$$\omega\backslash omega\boldsymbol{\omega}, where\ 0 < \omega \leq 10 < \backslash omega\backslash leq\ 1\mathbf{0} < \boldsymbol{\omega} \leq \mathbf{1}.$$

- Generate the sequence:

$$\{un(x)\}\backslash\{u_n(x)\backslash\}\{\boldsymbol{un}(\boldsymbol{x})\}$$ using the relaxed iterative formula.

- Continue the iteration until

$$\| un + 1 - un \| \infty < \varepsilon$$

or until the maximum number of iterations is reached.

- When the exact solution is available, compute the absolute error at selected grid points using

$$Ei = | uexact(xi) - uN(xi) |.$$

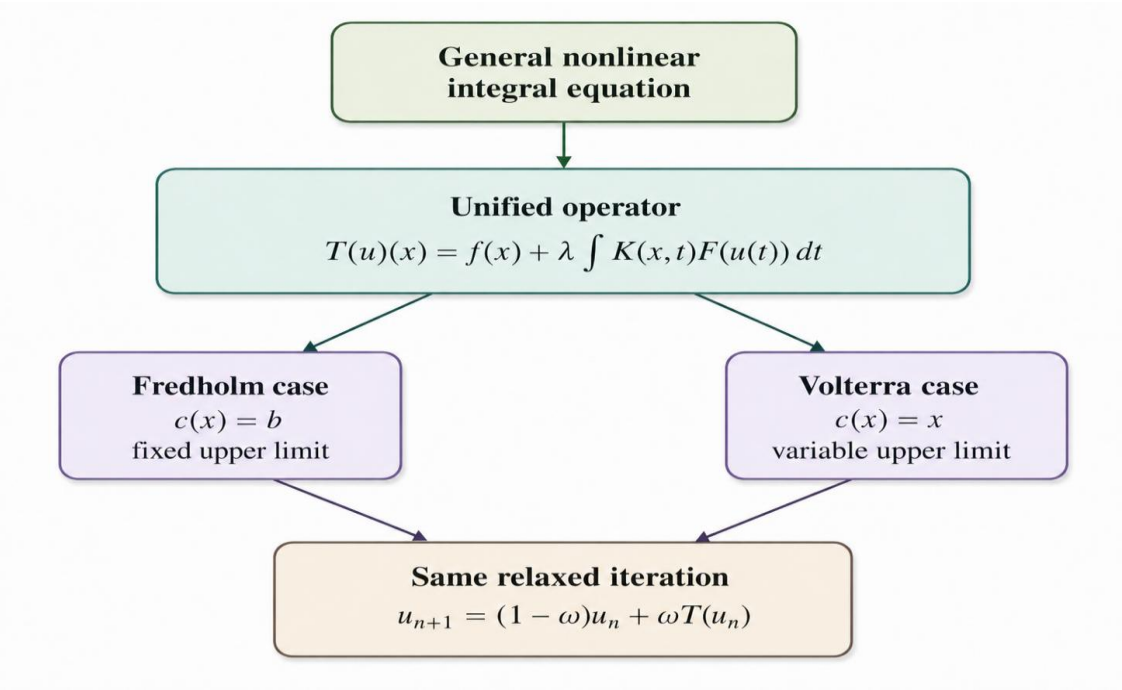


FIGURE 1. Unified Fredholm-Volterra operator diagram.

This diagram shows the paper's main mathematical idea. Instead of treating Fredholm and Volterra equations as completely separate problems, we write both using the same operator by changing the upper limit c(x). This supports the claim that the proposed iterative framework is general and flexible.

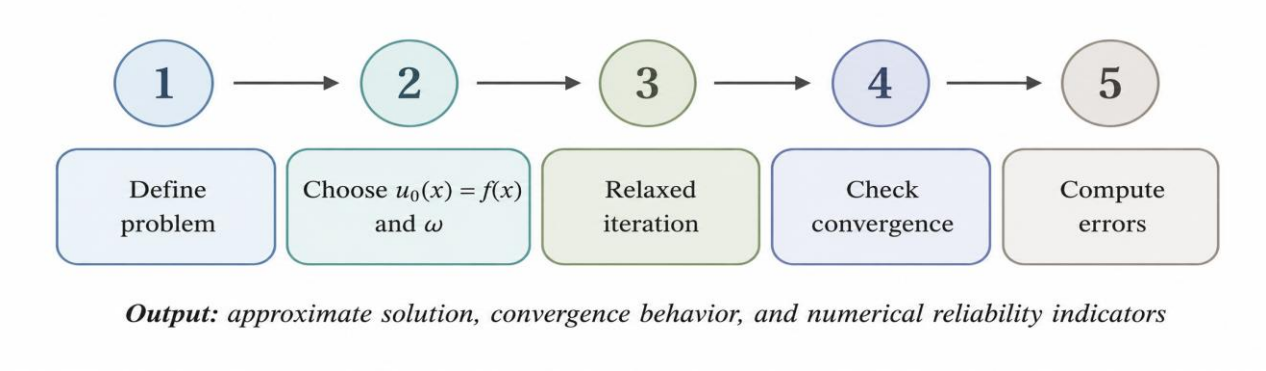


FIGURE 2. Computational workflow of the proposed generalized iterative framework.

The workflow makes the computational procedure clearer for readers. It starts from the model equation, passes through the relaxed iterative update, and ends with convergence checking and statistical error analysis. This connects the theoretical method to the numerical implementation.

V. CONVERGENCE ANALYSIS

The contraction mapping principle establishes the convergence of the proposed method. The analysis below is stated for the unified operator and applies to both the Fredholm and Volterra cases under the corresponding integration limits.

Assume that the kernel is bounded and that F is Lipschitz continuous with constant L:

$$|K(x,t)| \le M, \quad |F(u) - F(v)| \le L|u - v|. \quad (9)$$

For any two continuous functions u and v,

$$|(Tu)(x) - (Tv)(x)| \le |\lambda|ML(b - a)\|u - v\|_\infty. \quad (10)$$

Therefore, if

$$q = |\lambda|ML(b - a) < 1. \quad (11)$$

then T is a contraction on the considered function space. Consequently, T has a unique fixed point, and the sequence generated by the basic iterative scheme converges to the unique solution.

For the relaxed scheme, define

$$S(u) = (1 - \omega)u + \omega T(u). \quad (12)$$

The operator S satisfies

$$\|S(u) - S(v)\|_\infty \le [1 - \omega + \omega q]\|u - v\|_\infty. \quad (13)$$

Since $0 < \omega \le 1$ and $q < 1$, the factor $1 - \omega + \omega q$ is also less than one. Hence, the relaxed iteration is convergent under the same contraction setting. A standard error estimate for the basic method is

$$\|u - u_n\|_\infty \le \frac{q^n}{1 - q}\|u_1 - u_0\|_\infty. \quad (14)$$

This estimate shows that the error decreases as the number of iterations increases, provided the contraction condition holds.

VI. NUMERICAL EXAMPLES AND RESULTS

This section validates the proposed framework using three nonlinear integral equations with known exact solutions. We selected these examples to cover both Fredholm and Volterra equations and to test quadratic and cubic nonlinearities.

For the numerical tables, we use the initial approximation $u_0(x) = f(x)$, set the relaxation parameter to $\omega = 1$, and run $N = 10$ iterations. The absolute error is calculated as

$$E_i = |u_{exact}(x_i) - u_N(x_i)|. \quad (15)$$

Table 2. Summary of the selected test problems.

Example	Type	Nonlinearity	Exact solution	Purpose
1	Fredholm, second kind	$u(t)^2$	$u(x) = x$	Quadratic nonlinearity
2	Volterra, second kind	$u(t)^2$	$u(x) = \exp(x)$	Variable upper limit
3	Fredholm, second kind	$u(t)^3$	$u(x) = \sin(\pi x)$	Cubic nonlinearity

1. EXAMPLE 1: NONLINEAR FREDHOLM INTEGRAL EQUATION WITH QUADRATIC TERM

Consider the nonlinear Fredholm integral equation

$$u(x) = \frac{3}{4}x + \int_0^1 xt[u(t)]^2\,dt, \qquad 0 \le x \le 1. \quad (16)$$

The exact solution is

$$u(x) = x. \quad (17)$$

The iteration formula is

$$u_{n+1}(x) = \frac{3}{4}x + \int_0^1 xt[u_n(t)]^2\,dt. \quad (18)$$

This example is useful because the exact solution is simple and the error can be computed directly at standard grid points.

Table 3. Numerical results for example 1 after 10 iterations.

x	Exact solution	Approximate solution	Absolute error	Iterations
0.0	0.0000000000	0.0000000000	0.0000e+00	10
0.2	0.2000000000	0.1999616228	3.8377e-05	10
0.4	0.4000000000	0.3999232457	7.6754e-05	10
0.6	0.6000000000	0.5998848685	1.1513e-04	10
0.8	0.8000000000	0.7998464913	1.5351e-04	10
1.0	1.0000000000	0.9998081141	1.9189e-04	10

The results show that the approximate solution is very close to the exact solution. Since the solution is linear, the approximation error increases proportionally with x, but it remains small after ten iterations.

2. EXAMPLE 2: NONLINEAR VOLTERRA INTEGRAL EQUATION WITH QUADRATIC TERM

Consider the nonlinear Volterra integral equation

$$u(x) = e^x - \frac{e^{2x} - 1}{4} + \frac{x}{2} + \int_0^x (x - t)[u(t)]^2\,dt, \qquad 0 \le x \le 1. \quad (19)$$

The exact solution is

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$$u(x) = e^x. \quad (20)$$

The iteration formula is

$$u_{n+1}(x) = e^x - \frac{e^{2x} - 1}{4} + \frac{x}{2} + \int_0^x (x - t)[u_n(t)]^2\, dt. \quad (21)$$

This example demonstrates the method's applicability to Volterra equations where the upper limit of integration depends on x.

Table 4. Numerical results for Example 2 after 10 iterations.

x	Exact solution	Approximate solution	Absolute error	Iterations
0.0	1.0000000000	1.0000000000	0.0000e+00	10
0.2	1.2214027582	1.2214027582	3.9830e-11	10
0.4	1.4918246976	1.4918246976	4.1270e-11	10
0.6	1.8221188004	1.8221188004	9.4911e-12	10
0.8	2.2255409285	2.2255409285	7.5322e-12	10
1.0	2.7182818285	2.7182818284	5.9045e-11	10

The errors in this example are very small across the full interval. This indicates that the iterative framework performs well for the smooth exponential solution.

3. *EXAMPLE 3: NONLINEAR FREDHOLM INTEGRAL EQUATION WITH CUBIC TERM*

Consider the nonlinear Fredholm integral equation

$$u(x) = sin(\pi x) - \frac{2x}{3\pi} + \int_0^1 x(1 - t)[u(t)]^3\, dt, \qquad 0 \le x \le 1. \quad (22)$$

The exact solution is

$$u(x) = sin(\pi x). \quad (23)$$

The iteration formula is

$$u_{n+1}(x) = sin(\pi x) - \frac{2x}{3\pi} + \int_0^1 x(1 - t)[u_n(t)]^3\, dt. \quad (24)$$

This example tests the framework on a stronger nonlinear term than the quadratic cases.

Table 5. Numerical results for Example 3 after 10 iterations.

x	Exact solution	Approximate solution	Absolute error	Iterations
0.0	0.0000000000	0.0000000000	0.0000e+00	10
0.2	0.5877852523	0.5877847717	4.8059e-07	10
0.4	0.9510565163	0.9510555552	9.6110e-07	10
0.6	0.9510565163	0.9510550746	1.4417e-06	10
0.8	0.5877852523	0.5877833301	1.9222e-06	10
1.0	0.0000000000	-0.0000024028	2.4028e-06	10

The cubic example also shows good agreement with the exact solution. The largest error occurs near the endpoint x=1, but the value remains very small after ten iterations.

4. DISCUSSION OF NUMERICAL RESULTS

The numerical experiments show that the proposed relaxed iterative framework works for both Fredholm and Volterra nonlinear integral equations. We intentionally chose the three examples to test different behaviors: a quadratic Fredholm equation with a linear exact solution, a quadratic Volterra equation with an exponential exact solution, and a cubic Fredholm equation with a trigonometric exact solution. This variety strengthens the numerical validation because the same iterative formulation is applied without changing the basic structure of the method.

Example 1 shows a small, monotonic increase in the absolute error as x increases. This behavior is expected because the kernel contains the factor xt, which makes the integral contribution larger near the right end of the interval. Even so, the maximum error after ten iterations remains on the order of 10^-4, indicating stable convergence for the selected problem.

Example 2 gives the smallest errors among the three test problems. The Volterra structure naturally favors stepwise approximation because the value at x depends only on the interval [0,x]. In addition, the smooth exponential exact solution helps the iterative sequence converge rapidly, producing errors close to machine-level magnitude in the reported grid points.

Example 3 confirms that the framework remains reliable when the nonlinearity is increased from quadratic to cubic. Although the endpoint error is slightly larger than the interior errors, the maximum error remains very small. This suggests that the proposed scheme can handle stronger nonlinear terms when the contraction requirements are reasonably satisfied.

Overall, the results agree with the theoretical convergence analysis. The statistical indicators support the same conclusion: the mean absolute error, RMSE, and maximum error remain low for all examples. Therefore, the method is not only mathematically justified under contraction assumptions but also practically accurate for representative nonlinear integral equations of the second kind.

5. STATISTICAL ERROR ANALYSIS AND VISUALIZATION

To strengthen the numerical discussion, we processed the absolute error values from the three test problems using statistical calculations and graphical visualization. The statistical indicators used were the mean absolute error (MAE), maximum absolute error, root mean square error (RMSE), and standard deviation (SD). These measures provide a clearer view of average accuracy, worst-case error, and error spread across the selected grid points.

Table 6. Statistical summary of absolute errors after 10 iterations.

Example	MAE	Maximum error	RMSE	Standard deviation	Short interpretation
Example 1	9.5943e-05	1.9189e-04	1.1619e-04	7.1798e-05	Small error; largest among the three because the final error grows with x.
Example 2	2.6195e-11	5.9045e-11	3.3968e-11	2.3689e-11	Very high accuracy; errors are close to machine-level values.
Example 3	1.2014e-06	2.4028e-06	1.4550e-06	8.9904e-07	Stable result for cubic nonlinearity with very small endpoint error.

The statistical summary confirms the accuracy observed in the numerical tables. Example 2 gives the smallest error values because the Volterra structure and the smooth exponential exact solution produce very rapid convergence. Example 1 has the largest maximum error, but it remains small after ten iterations. Example 3 shows that the method remains reliable even for a cubic nonlinear term.

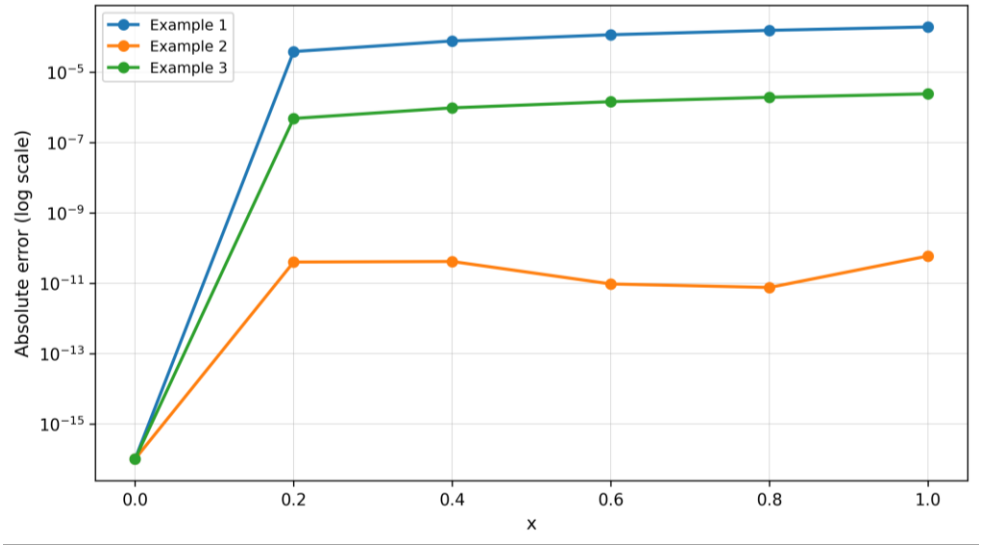


FIGURE 3. Absolute error distribution for the three numerical examples after 10 iterations.

The error curves show that all examples remain accurate on the full interval [0,1]. The first example shows a gradually increasing error as x increases, while the second example has the smallest errors overall. The third example has slightly larger errors near the endpoint but still remains highly accurate.

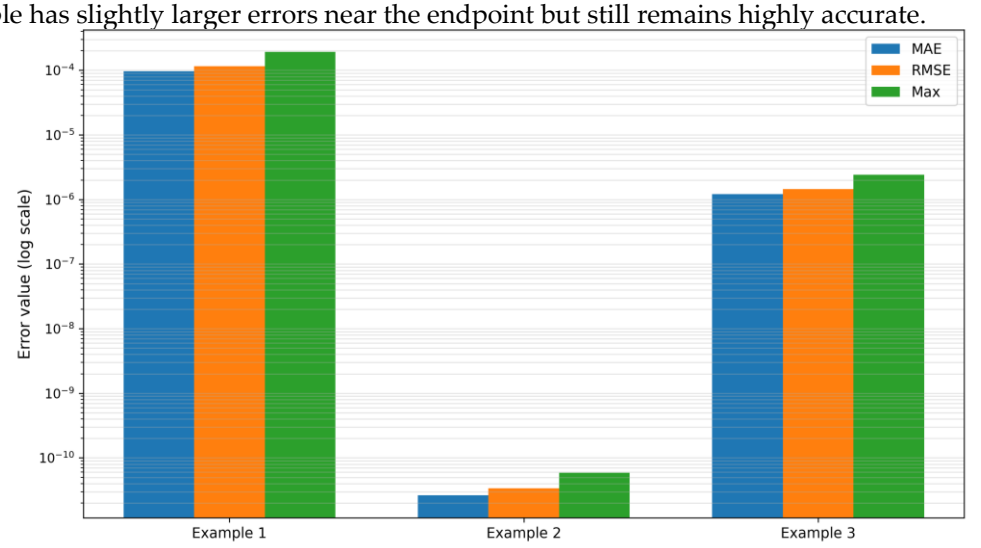


FIGURE 4. Statistical comparison of MAE, RMSE, and maximum absolute error.

This bar chart compactly compares the three examples using standard statistical indicators. We use a logarithmic scale because the errors differ by several orders of magnitude. The results show that the proposed method performs especially well for the Volterra example and remains stable for the nonlinear Fredholm examples.

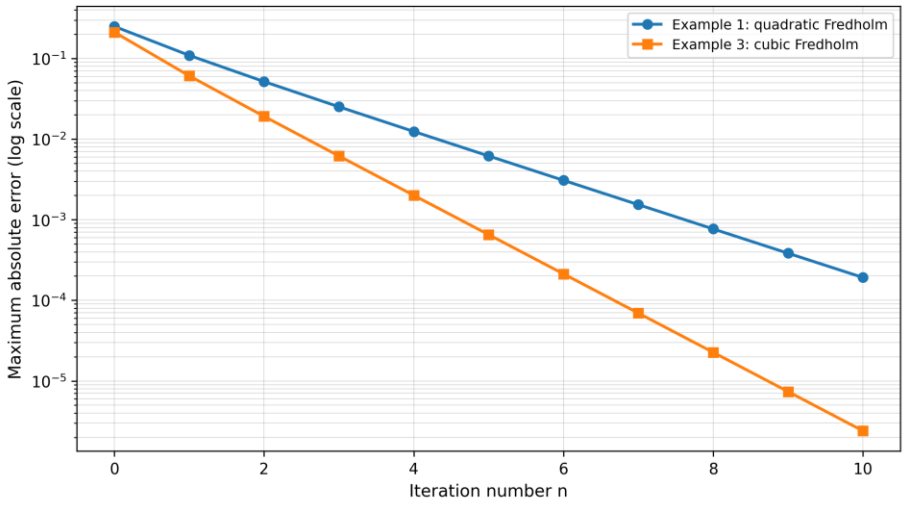


FIGURE 5. Convergence behavior of the Fredholm examples across iterations.

The convergence curves demonstrate how the maximum absolute error decreases as the number of iterations increases. Both curves decline steadily, supporting the convergence argument in Section 5. The cubic Fredholm example starts with a large error because of its stronger nonlinearity, but it still converges to a very small error after ten iterations.

VII. COMPARISON WITH EXISTING METHODS

This qualitative comparison is supported by the literature review, the numerical examples, and the statistical analysis in the results section. It explains how the proposed framework relates to common solution strategies for nonlinear integral equations.

Table 7. Qualitative comparison with common methods.

Method	Main idea	Strength	Limitation
Successive approximation	Uses repeated substitution in the integral operator.	Simple and direct.	May converge slowly or require strong contraction.
Newton-type iteration	Linearizes the nonlinear operator around the current approximation.	Often fast near the solution.	Needs derivative information and more computation.
Adomian decomposition	Represents the solution and nonlinear term using series components.	Useful for analytic approximations.	Algebra may become complicated for high-order terms.
Homotopy perturbation	Builds a homotopy and expands the solution in a perturbation series.	Flexible for many nonlinear models.	Accuracy depends on series truncation.
Variational iteration	Constructs correction functionals using Lagrange multipliers.	Can provide accurate semi-analytic forms.	Multiplier construction may not be straightforward.

Method	Main idea	Strength	Limitation
Proposed framework	Uses a unified iterative operator with optional relaxation.	Applicable to Fredholm and Volterra equations; stable under contraction conditions.	Relaxation parameter and discretization must be chosen carefully.

VIII. CONCLUSION

This study presented a generalized iterative framework for solving nonlinear integral equations of the second kind. The main contribution is a unified operator formulation that covers both Fredholm equations with fixed integration limits and Volterra equations with variable upper limits. By introducing a relaxation parameter into the classical successive approximation process, the method becomes more flexible while preserving a simple computational structure.

The convergence analysis showed that the proposed framework converges under standard bounded-kernel and Lipschitz-continuity assumptions when the associated nonlinear integral operator is a contraction. The relaxed operator also remained contractive under the same setting, supporting the approach's theoretical reliability.

The numerical results confirmed the method's effectiveness on three representative test problems. The method produced accurate approximations for Fredholm and Volterra equations and remained stable for both quadratic and cubic nonlinearities. The statistical error measures further supported the results by showing low mean absolute error, low RMSE, and a small maximum error across the selected grid points.

Compared with methods that require complex algebraic transformations, derivative information, or problem-specific constructions, the proposed framework is easy to implement and adaptable to different forms of nonlinear integral equations. Its main advantage is that it combines the simplicity of successive approximation with the additional stability control offered by the relaxation parameter.

The study is limited by the number of benchmark problems and by the qualitative comparison with existing methods. Future work should therefore include quantitative comparisons with established methods, testing different relaxation parameters, extending to systems of nonlinear integral equations, and applying the framework to problems with more complex kernels or weak singularities.

IX. FUTURE WORK

- Test different values of the relaxation parameter ω and compare convergence rates.
- Apply the framework to additional nonlinear functions F(u), including exponential and trigonometric nonlinearities.
- Compare the proposed method quantitatively with Adomian decomposition, homotopy perturbation, variational iteration, and Newton-type methods.
- Extend the method to systems of nonlinear integral equations.

Author Contributions

The author conducted the conceptualization, methodology, data analysis, investigation, writing, review, editing, and final approval of the manuscript.

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Data Availability

The dataset will be available from the author upon reasonable request.

Conflicts of Interest

The author declares no conflict of interest.

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